Asia-Pacific Financial Markets **6:** 253–273, 1999.

An Empirical Analysis on Log-Utility Asset Management

HIROSHI ISHIJIMA
Faculty of Policy Management, Keio University, 5322 Endoh, Fujisawa, 252-8520, Japan, e-mail: hi@sfc.keio.ac.jp, hiroshi_ishijima@hi-ho.ne.jp

Abstract. In this paper, we empirically verify the optimal portfolio schemes for the log-utility investor under incomplete information which converge to the optimal portfolio maximizing the expected log-utility under complete information. That is, our main interest lies in examining whether these schemes really attain the above desired properties, in the NYSE/AMEX stock market. With these properties regarded as performance measures, our empirical research is executed through a sensitivity analysis with transaction costs. Moreover, we show the interesting character of the U.S. stock market exhibited through the analysis.

Key words: portfolio selection, log-utility, learning schemes, transaction costs, NYSE/AMEX stock market.

1. Introduction

The application of modern portfolio theory has been developed with the models which maximize the expected value of von Neumann-Morgenstern utility. Among these, the application of log-utility can be one possible scheme for reinvesting investors' wealth over the long run. Preceding empirical works concerned with log-utility and power-utility asset management have been published by several authors (Grauer and Hakansson, 1982, 1985, 1986, 1987). In these papers, the portfolio composition and the ex post geometric mean and variance are examined by varying risk attitude, i.e. the Arrow-Pratt relative risk aversion. But these empirical analyses were not based on a scheme for how to apply these portfolio theories constructed by investors' expected utility under incomplete information. In this paper, using the several schemes to construct the portfolio which converges to the one maximizing the expected log-utility (ExLog portfolio), we verify whether these schemes can provide the maximal ex post log-utility in the practical U.S. stock market. The above schemes, given in Ishijima and Shirakawa (1999), are based on the continuous-time framework and require several parameters for their application. However, these parameters, such as the learning period, the security universe, and so on, can not be determined theoretically. Hence, our empirical research is executed using a sensitivity analysis. That is, by varying one specified parameter with all the other parameters fixed, we examine the sensitivity of the ex post log-utility for each learning scheme. Also we carry out these analyses under the existence of transaction costs as in the practical market environment. Since our treatment for transaction costs is rather new in that costs are incurred for each asset's rebalance level, it is applicable in the actual market environment. In Section 2, the model for log-utility investors is addressed. Then we summarize the properties of two schemes employed in this empirical analysis and propose the algorithms for these schemes. That is, these are expected to asymptotically learn the portfolio maximizing the expected log-utility under complete information, and to myopically derive the optimal portfolio under incomplete information. In Section 3, the method of the empirical analysis is explained. That is, investment rules, parameters, and portfolio performance measures are described. In Section 4, the results of the empirical analysis are given in a sensitivity analysis format. That is, the results are presented by varying one specified parameter with all the other parameters held fixed. Finally, the implications of these results are explored.

2. The Objective of Log-Utility Investors and Its Learning Schemes

2.1. THE MODEL

First we define the objective of log-utility investors. We consider the market in which there exists n assets, and their prices $S_{i,t}$ $(i = 1, \ldots, n)$ are generated by the following dynamics:

$$S_{i,t+1} = S_{i,t}\left(1 + \mu_{i,t+1} + Z_{i,t+1}\right), \quad 0 \le t \le T - 1 ,$$

where $Z_{i,t+1}$ is some martingale increment process and $\mu_{i,t+1}$ is predictable increment part. That is, the filtration is given by $\mathcal{F}_t \stackrel{\triangle}{=} \sigma(\boldsymbol{Z}_u; u = 1, \ldots, t) \vee \mathcal{F}_0$ and $\mu_{i,t+1}$ is $\mathcal{F}_t$-adapted. Also we treat the log-utility type investor, such that

$$u(x) \stackrel{\triangle}{=} \log x \ (x > 0) .$$

Suppose that the investors rebalance their portfolios in discrete time intervals, subject to the budget constraint and the nonnegativity requirements for the portfolio weights. That is, investors select their portfolios $\boldsymbol{b}$ in the simplex $\mathbf{D}$:

$$\mathbf{D} \stackrel{\triangle}{=} \left\{\boldsymbol{b} \in \mathbf{R}^n \,\middle|\, \boldsymbol{b}'\mathbf{1} = 1,\ \boldsymbol{b} \ge \mathbf{0}\right\} . \tag{1}$$

Without transaction costs, the investors' wealth V_t reinvested until time t is given as

$$V_t = V_{t-1}\boldsymbol{b}_t'\boldsymbol{X}_t = V_0 \prod_{u=1}^{t} \boldsymbol{b}_u'\boldsymbol{X}_u ,$$

where $V_0 = 1$, $\boldsymbol{X}_t = \left(X_{1,t}, \ldots, X_{n,t}\right)'$ and $X_{i,t} \stackrel{\triangle}{=} S_{i,t}/S_{i,t-1}$ denotes price-relatives of assets. Under these conditions, with the terminal-time T provided, the investors' terminal log-utility maximization problem is stated as follows:

$$\mathbf{P}_0 \left| \begin{array}{ll} \underset{\mathbf{b}_\bullet}{\text{maximize}} & E[u(V_T)] \\ \text{subject to} & \boldsymbol{b}_t \text{ is } \mathcal{F}_t - \text{predictable}\ , \\ & \boldsymbol{b}_t \in \mathbf{D}\ . \end{array} \right.$$

Let the solution for this problem $\mathbf{P}_0$ be $\boldsymbol{b}_\bullet^*$ and we call it the *$\mathcal{F}_t$-predictable portfolio maximizing the expected value of the log-utility (ExLog portfolio).*

Then our question emerges. The log-utility investors can only observe a realization, $\boldsymbol{X}_t$, of the underlying stochastic process. In other words, they are provided with the information $\mathcal{G}_t \overset{\triangle}{=} \sigma(\boldsymbol{X}_u; u = 1, \ldots, t)$, and note that $\mathcal{G}_t \subset \mathcal{F}_t$. Our main interest lies in learning the $\mathcal{F}_t$-predictable portfolio $\boldsymbol{b}_\bullet^*$ in the practical market. Moreover, we empirically search for the optimal portfolio under $\mathcal{G}_t$ for the log-utility investor. To do this, we employ two schemes proposed by Ishijima and Shirakawa (1999) and Shirakawa and Ishijima (1997), and we summarize these in the following two subsections.

2.2. THE SPOP SCHEME

According to the theory discussed in Ishijima and Shirakawa (1999), the following scheme assures the log-utility investors of learning the optimal portfolio under complete information in the long run. Also, it derives the optimal portfolio under incomplete information for the log-utility investor. The assumption in their study is that n asset prices follow geometric Brownian motion:

$$\mathrm{d}\boldsymbol{S}_t = \mathrm{diag}(\boldsymbol{S}_t)\,(\boldsymbol{\mu}\mathrm{d}t + \Sigma \mathrm{d}\mathbf{W}_t)\ , \tag{2}$$

where $\mathrm{diag}(\boldsymbol{S}_t)$ is a diagonal matrix whose element is $\boldsymbol{S}_t$. Here, $\mathbf{W}_t = (W_{1t}, \ldots, W_{nt})'$ denotes an n-dimensional standard Brownian motion and the filtration $\tilde{\mathcal{F}}_t$ is generated by $\sigma\,(\mathbf{W}_u; 0 \le u \le t)$ and $\tilde{\mathcal{F}}_0$. We assume that $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)'$ is an $\tilde{\mathcal{F}}_0$-measurable random vector following multivariate normal distribution and $\Sigma = \left(\sigma_{ij}\right)_{1\le i,j\le n}$ is a constant diffusion parameter. Also, it is supposed that $\{\mathbf{W}_t;\ t \ge 0\}$ and $\boldsymbol{\mu}$ are independent. Here, we note that if the log-utility investors completely know the asset prices follow the above process (2), their ExLog portfolio is constant through investment horizons, i.e., $\boldsymbol{b}_t^* \equiv \boldsymbol{b}^*$.

In the actual market environment, however, investors can have the following class of information.

Information 1 *Incomplete information: Investors have no prior distribution information for the drift parameter $\boldsymbol{\mu}$. That is, the differential entropy of the prior distribution for the drift parameter is asymptotically infinite. And they are only provided with the information $\tilde{\mathcal{G}}_t \subset \tilde{\mathcal{F}}_t$ generated by a realized asset price process of Equation (2) as follows:*

$$\tilde{\mathcal{G}}_t \overset{\triangle}{=} \sigma\,(\boldsymbol{S}_u; 0 \le u \le t)\ . \tag{3}$$

Under the settings above, Ishijima and Shirakawa (1999) proposed the scheme called *sample path-wise optimal portfolios (SPOP).* In this scheme, by using constant portfolios, the sample path-wise wealth $V_t(\boldsymbol{b})$ is obtained. Then using the $\tilde{\mathcal{G}}_t$-predictable portfolio maximizing $V_t(\boldsymbol{b})$, the log-utility investors can attain the $\tilde{\mathcal{F}}_t$-predictable ExLog portfolio $\boldsymbol{b}^*$, which is the solution of $\mathbf{P}_0$, in the long run. Moreover, for the log-utility investors, the SPOP is optimal under $\tilde{\mathcal{G}}_t$ which is guaranteed by the continuous Bayesian updating scheme with the prior distribution for $\boldsymbol{\mu}$ being endowed with asymptotically infinite differential entropy (Ishijima and Shirakawa, 1999).

We can expect the SPOP may have the above two properties in the actual market. Then the discrete-time version of SPOP is given by the following problem:

$$\mathbf{P}_1(t)\ \left|\ \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & V_t(\boldsymbol{b}) = \prod_{u=t-L+1}^{t} \left(\boldsymbol{b}'\boldsymbol{X}_u\right) \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\ . \end{array}\right.$$

Here L is some period in which the SPOP is expected to learn(estimate) constant ExLog portfolios $\boldsymbol{b}^*$ and to derive the optimal portfolio under $\mathcal{G}_t$ for the log-utility investors, and we call L *learning periods*. Note that the operation product for portfolio value V_t should start from $u = 1$ to be accommodated by the theory. However, the reason why $u = t-L+1$ is used instead of $u = 1$ is that the log-utility investors may use shorter information set $\mathcal{G}_{t-L+1}^{t} \overset{\triangle}{=} \sigma\left(\boldsymbol{X}_u;\, u = t - L + 1, \ldots, t\right)$ rather than $\mathcal{G}_t$, to derive the optimal portfolio. Problem $\mathbf{P}_1(t)$ is equivalent to the following problem:

$$\mathbf{P}_1'(t)\ \left|\ \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & g(\boldsymbol{b}) = \frac{1}{L}\sum_{u=t-L+1}^{t} \log \boldsymbol{b}'\boldsymbol{X}_u \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\ . \end{array}\right.$$

Next, we describe the algorithm for searching for the SPOP. Cover (1984) proposed its iterative algorithm as follows:

(Algorithm for the SPOP)

The iteration $\boldsymbol{b}^{(j)}$ is defined as

$$\left|\ \begin{array}{ll} b_i^{(j+1)} = \frac{b_i^{(j)}}{L}\sum_{u=t-L+1}^{t} \frac{X_{i,u}}{\boldsymbol{b}^{(j)'}\mathbf{X}_u}\ , & \\ b_i^{(0)} = \frac{1}{n}\ , & i = 1, \ldots, n\ . \end{array}\right. \qquad (4)$$

2.3. THE UP SCHEME

The second scheme, having two properties as the SPOP, is to use *universal portfolios* (UP) (Cover, 1991; Cover and Ordentlich, 1996; Shirakawa and Ishijima, 1997). The UP is defined, by using the sample path-wise value $V_t(\boldsymbol{b})$, as follows:

$$\boldsymbol{b}_t^{\sharp} \stackrel{\triangle}{=} \int_{\boldsymbol{b}\in\mathbf{A}} \boldsymbol{b} f_t(\boldsymbol{b}) \mathrm{d}\boldsymbol{b}\,, \tag{5}$$

where

$$f_t(\boldsymbol{b}) \stackrel{\triangle}{=} \frac{V_t(\boldsymbol{b})}{\int_{\boldsymbol{b}\in\mathbf{A}} V_t(\boldsymbol{b})\mathrm{d}\boldsymbol{b}} \tag{6}$$

is the weighting density function of constant portfolios. With this definition such that the integral for $\boldsymbol{b}$ is taken over $\mathbf{A} \stackrel{\triangle}{=} \left\{\boldsymbol{b} \,\middle|\, \boldsymbol{b} \in \mathbf{R}^n \right\}$, the UP, as well as the SPOP, is optimal for the log-investors under $\tilde{\mathcal{G}}_t$. Moreover, even if the integral range for $\boldsymbol{b}$ is restricted to be $\mathbf{A} \stackrel{\triangle}{=} \mathbf{D}$ of (1), the UP, as well as the SPOP, can learn $\boldsymbol{b}^*$ asymptotically. Next, an algorithm for the UP is explained. Discrete sampling expression of the UP, using learning periods L, is stated as

$$b_{i,t}^{\sharp} \stackrel{\triangle}{=} \sum_{k=1}^{N} b_i^{(k)} f_t(\boldsymbol{b}^{(k)}) = \sum_{k=1}^{N} b_i^{(k)} \frac{\prod_{u=t-L+1}^{t} \boldsymbol{b}^{(k)'} \boldsymbol{X}_u}{\sum_{k=1}^{N} \left(\prod_{u=t-L+1}^{t} \boldsymbol{b}^{(k)'} \boldsymbol{X}_u\right)}\,, \tag{7}$$

where N is the size of samplings for constant portfolios, and $\boldsymbol{b}^{(k)}$ is the kth sampled constant portfolio. Universality is possessed in the weighting density function $f_t(\boldsymbol{b})$. This function shows us how much each constant portfolio grew in the past. And according to $f_t(\boldsymbol{b})$, the UP determines its weights to ride this “winning horse”, which almost surely becomes the ExLog portfolio $\boldsymbol{b}^*$.

The only challenge for the algorithm for the UP is how to sample constant portfolios $\boldsymbol{b}$ efficiently. Here we propose the roughest $(n+1)$-sampling, that is n extreme points and an equally weighted portfolio:

$$\begin{aligned} \boldsymbol{b} &= \left\{\boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(k)}, \ldots, \boldsymbol{b}^{(n+1)}\right\} \\ &= \left\{(1, 0, \ldots, 0)', (0, 1, 0, \ldots, 0)', \ldots, (0, \ldots, 0, 1)', (\frac{1}{n}, \ldots, \frac{1}{n})'\right\}\,. \end{aligned}$$

Using this constant portfolio sampling, an algorithm for the UP is given as follows:

(Algorithm for the UP)

The UP at time t is :

(At $t = 0$) Define $b_{i,0}^{\sharp} = 1/n$ $(i = 1, \ldots, n)$.

(At $t > 0$) Calculate $b_{i,t}^{\sharp}$ $(i = 1, \ldots, n)$ of (7) using sampling $\boldsymbol{b} = \left\{\boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(n+1)}\right\}$.

2.4. TRANSACTION COSTS

Since transaction costs are not treated in the analysis of Ishijima and Shirakawa (1999), both the SPOP and UP in the discrete-time framework follow them. The reason is that it is difficult to show two properties of both the SPOP and UP, with transaction costs taken into account.

In the actual market, however, transaction costs are incurred every time investors rebalance. Hence we consider them when we empirically evaluate the performance of portfolios. If investors allocate their wealth V_{t-1} via the portfolio $\boldsymbol{b}_{t-1}$ at the beginning of the period t, the wealth becomes $V_t = V_{t-1}\boldsymbol{b}'_{t-1}\boldsymbol{X}_t$, without costs. But if we introduce the linear cost rates γ for the rebalance amount, the wealth becomes

$$V_t = V_{t-1}\frac{\sum_{i=1}^{n}\frac{b_{i,t-1}X_{i,t}}{1-\gamma\zeta_{i,t}}}{\sum_{j=1}^{n}\frac{b_{j,t}}{1-\gamma\zeta_{j,t}}}\,, \tag{8}$$

where $\zeta_{i,t} \triangleq 1 \ \left(\text{if } b_{i,t} \geq \kappa_t b_{i,t}^{-}\right)$, $\zeta_{i,t} \triangleq -1 \ \left(\text{if } b_{i,t} < \kappa_t b_{i,t}^{-}\right)$, $b_{i,t}$ is the portfolio weight for the ith asset at the beginning of the period $t+1$, and $b_{i,t}^{-}$ is the weight (ratio) of the ith asset value relative to the value of portfolio at the end of the period t, i.e., $b_{i,t}^{-} \triangleq \dfrac{b_{i,t-1}X_{i,t}}{\boldsymbol{b}'_{t-1}\boldsymbol{X}_t}$. The adjustment coefficient κ_t is the fixed point obtained from the following equation:

$$\kappa_t = \frac{\sum_{i=1}^{n} b_{i,t-1}X_{i,t}\cdot\sum_{j=1}^{n}\frac{b_{j,t}}{1-\gamma\zeta_{j,t}}}{\sum_{l=1}^{n}\frac{b_{l,t-1}X_{l,t}}{1-\gamma\zeta_{l,t}}}\,. \tag{9}$$

The derivation of above formulae (8) and (9) is given in Appendix A.

3. Method of Empirical Analysis

We employed the theory of Ishijima and Shirakawa (1999), and we require some settings to apply these theories. That is, we still require some rules, input parameters, and portfolio performance measures which cannot be determined by the theories. Hence, we empirically verified the theories in a sensitivity analysis-like method.

3.1. INVESTMENT RULES

The log-utility investors reinvest their wealth only in the NYSE and AMEX stock markets according to the following rules.

1. Rebalancing: Investors rebalance their wealth according to their optimal portfolios at the beginning of each month.

2. Portfolios: Log-utility investors have two schemes, the SPOP and the UP, to determine their portfolios which are expected to learn the ExLog portfolio $\boldsymbol{b}^*$ in the long run and to derive the optimal portfolio under $\mathcal{G}_t$. Also we study the equally weighted portfolio (EP) for benchmark. We assume that investors start with the EP at the beginning of the performance measurement horizon, regardless of their decision criteria.
3. Information: At t, investors only have incomplete information $\mathcal{G}_t$, which is the rates of returns. These stock returns are obtained from the CRSP CD-ROM, so every stock price and dividend are adjusted at each period. At decision point t, the information from L months past to the last month of t, i.e. $\{\boldsymbol{X}_{t-L+1}, \ldots, \boldsymbol{X}_t\}$ is available for investors. For our purposes, the entire universe of the market is composed of 749 stocks, whose rates of returns have been observed in full from January 1971 up to December 1995, i.e. 300 months.
4. Transaction costs, divisibility, and budget constraints: Investors incur costs for every rebalance. We assume investors' wealth is infinitely divisible. Also investments are made within the investors' own wealth under no short-selling, i.e. within the simplex **D** of (1).

3.2. PARAMETERS

When investors reinvest their wealth in the stock market, a few elements may affect the ex post mean of their log-utility, besides the choice of portfolios. These elements are listed below.

1. Learning periods L : The log-utility investors use L periods to derive the optimal portfolio extracting from shorter $\mathcal{G}_t$. We set L as $L = 3, 12, 36, 60, 120$ months, and as all the past months which is available at each portfolio decision point t.
2. Security universe n : From the mathematical viewpoint, the larger the investment universe gets, the higher investors can expect their objectives to increase. But asset choice may have an ex post affect. So we vary the data-set in several cases. Ordering from small to large, we prepare the 27 securities composing the DJIA, the 74 securities composing the S & P 100, the 274 securities composing the S & P 500, and the 749 securities of the entire market universe.
3. Transaction cost rates γ : These are explained at Subsection 2.4, and we set cost rates as $\gamma = 0,\ 1,\ 2$ (%) in the analysis.

3.3. PORTFOLIO PERFORMANCE MEASURES

Under above rules and parameters, we measure the performance of three portfolios. We set the measurement horizon from January 1993 to December 1995. At January 1993, the initial point in the measurement horizon, every investor has the EP. The measures we use in the analysis are as follows.

1. Ex post mean of investors' log-utility: The gap in the expected log-utility base is shown in Shirakawa and Ishijima (1997), among the ExLog portfolio, the SPOP, and the UP. It is proved that both the SPOP and UP converge to the ExLog portfolio $\boldsymbol{b}^*$ and so their asymptotic growth rates, i.e. $\lim_{T\to\infty} \frac{1}{T} \log(V_T)$ are the same. Following the above results, we empirically investigate the ex post mean of the log-utility at finite measurement periods of the portfolio performance.
2. Turnover: This is the measure revealing how much the portfolio changes its weights. This is defined as the ratio of the amounts of rebalances, being summed up for all securities in absolute value, relative to the entire portfolio value before rebalancing. Derivation and details are placed in Appendix A.
3. The Kullback-Leibler distance: The Kullback-Leibler distance (KL distance) measures the divergence of the portfolio from the EP in our analysis. This is an information number and satisfies the axioms of distance. The KL distance from the EP, $\boldsymbol{b}^n = (1/n, \ldots, 1/n)'$ to any portfolio $\boldsymbol{b}$ is defined as follows:

$$D(\boldsymbol{b}\|\boldsymbol{b}^n) \triangleq \sum_{i=1}^{n} b_i \log \frac{b_i}{b_i^n} = \sum_{i=1}^{n} b_i (\log n + \log b_i) = \log n - H(\boldsymbol{b}),$$

where $H(\boldsymbol{b}) = \sum_{i=1}^{n} b_i(-\log b_i)$ is entropy and this is nonnegative. One reason for using this distance is that this number is bounded, that is: $0 \leq D(\boldsymbol{b}\|\boldsymbol{b}^n) \leq \log n$. The lower bound is attained when $\boldsymbol{b} = \boldsymbol{b}^n$ and the upper bound is attained when $\boldsymbol{b} = \{(1, 0, \ldots, 0)', \ldots, (0, \ldots, 0, 1)'\}$.

Now we proceed to the results.

4. Results in the U.S. Stock Market

4.1. VARYING LEARNING PERIODS L

First, fix the number of stocks and the transaction costs to be $n = 74$ (composing the S & P 100) and $\gamma = 1\%$, respectively. The results for varying the learning periods are given in Table I, and Figure 1. The result here represents the main finding and the implication of this empirical analysis, and subsequent results are given to support them.

Both the SPOP and the UP rarely exceed the EP in the ex post mean of the log-utility, with any leaning periods. The shape of the ex post utility measured against the learning periods L is U-shaped, for both the SPOP and the UP. But a distinct difference between the two is the robustness in learning periods. As we can see in the turnover in Table I, the SPOP is not able to learn long-living portfolios consistently, except $L = ALL$. Then it changes its weights frequently. To make matters worse, the ex post utility of the SPOP varies among learning periods. From the KL distance given in Table I, the UP is to learn the portfolio near the EP, regardless of learning periods. So from a viewpoint of learning periods, the UP is a better learning scheme than the SPOP.

To consider the U-shape of the ex post utility, let us pay attention to the shortest and the longest learning periods. Slight as it is, the excess ex post utility compared to the EP can be seen for both the SPOP and the UP with the longest learning periods ($L = ALL$). With the shortest learning periods ($L = 3$), the ex post utility is at the same level. These two similar results imply quite different meanings.

With $L = ALL$, the KL distance, which is bounded by $\log n (\simeq 4.3)$, shows both the SPOP and the UP are near to the EP. This means that after long enough learning periods, both the SPOP and the UP get nearer to the portfolio which is close to the EP. And both the SPOP and the UP with longest learning periods keep their turnover low compared to those with the other learning periods (Table I). This is because these portfolios do not have to change their weights drastically, for the chosen portfolios will be good ones as they were in the past. Then, we can say that both the SPOP and the UP with enough learning are roughly constant as the ideal ExLog portfolio is supposed to be. This finding implies that the two portfolios, after long enough learning periods, converge to the portfolios which duplicate the ExLog portfolio. Also, these duplicated portfolios are near the EP in the KL distance sense and achieve almost the same highest ex post utility as the EP does. Furthermore, the results in the continuous-time framework, given by Ishijima and Shirakawa (1999), assert that both the SPOP and the UP are asymptotically optimal for the log-utility investors under incomplete Information 1. Then, these observations imply that the EP can be roughly regarded as the market portfolio for the log-utility investors. That's why most fund managers rarely outperform the EP in the practical market where transaction costs exist, recalling that the log-utility investor aims for growth maximization.

Then turning to the result with $L = 3$, the SPOP is quite different from the EP in the KL distance sense. This is because the concave optimization is executed with few states of returns relative to the number of stocks, and its optimal portfolio is composed of only a few stocks having positive weights.

But the UP is quite close to the EP. This is from the modestly adaptive property of the UP. Since the UP starting from the EP is intended to become the "winning horse", it requires longer price-relative sequences. Hence, with shorter learning periods such as three months, the additional learning from the EP is not incorporated into the UP. Since the EP is close to the ExLog portfolio under $L = ALL$, the consequence that both the EP and the UP achieve the same utility level is reasonable. While the SPOP is quite different from the EP and UP, it does yield the same performance. This is because extracting a portfolio in maximal growth from shorter information set yields the volatile portfolio weights. This leads to the poor utility level due to transaction costs. This is exhibited in the turnover in Table I. In the SPOP, the turnover is relatively high compared to the other portfolios, except those with the longest learning periods. Hence, the SPOP with shorter learning periods has a potential to provide a prominent utility compared to the duplicated ExLog portfolios, which are equivalent to the SPOP and the UP with all the available information, and which are near the EP. But that potential heavily depends on the set of price-relative vectors, judged from Figure 1 and the first column of Table I. The SPOP with shorter learning periods has a possibility to achieve the same or higher utility only when the investor composes the SPOP using a set of price-relative vectors which affect the next holding period's realization sufficiently enough to reverse the disadvantage in transaction costs.

According to the above results with shortest and longest learning periods, we can say that the general character of the U.S. stock market is such that extremely short or long sequences of returns in the past reflect worthwhile information for the log-utility investor and affect the ex post portfolio performance. That is, investors who concentrate on a series of short term play-offs, and investors who are equally allocating their wealth to blue chip stocks for a long time, both get rational utility provided by the EP. These findings can be linked with the concept, which is used in explaining stock price behavior, called "mean reversion" (Fama and French, 1988; Poterba and Summers, 1988). There seems to exist permanent and worthwhile information in stock prices for the log-utility investor, but some temporary information may overwhelm it. Investors can observe prices, but they are unable to separately observe the original information summed up to the prices. Reviewing our results, longest learning periods enable the log-utility investors to pick up the permanent components. Extra ex post utility with shortest L is provided by the transitory components.

4.2. VARYING SECURITY UNIVERSE n

The results with $L = ALL$ and $L = 3$ are given in Table II and Table III, respectively. With $L = ALL$, from Table II: as the asset universe gets smaller, all three portfolios increase their ex post utility, except when $n = 295$. This trend shows that the effect of asset choice exists. That is, the EP composed of selected stocks in an elitism course outperforms the EP of the entire market. In every asset universe except $n = 74$, the EP performs best. And as the universe gets larger, the KL distance of the SPOP and the UP gets larger; and the ex post utility becomes worse. This trend is most striking in the SPOP, which is conjectured as follows. As we pointed out in the result by varying learning periods, when the concave optimization is executed with few states of returns relative to the number of stocks, its optimal portfolio is composed of only a few stocks having positive weights. So even if investors use the full 299 months of learning periods, the SPOP for the next holding period is at most composed of 300 stocks having positive weights. Then the larger the universe gets, the more divergent the SPOP becomes from the EP.

Then proceed to $L = 3$. From Table III, different results are obtained for the SPOP. As asset universes get larger, the ex post utility of the SPOP increases, except when $n = 295$. When $n = 749$, for the first time, a portfolio clearly outperforms the EP in the ex post utility. Turnover above 100% (Table III) tell us too much transaction costs are being incurred. As we stated earlier, it can be said that transitory but worthwhile information exists sufficient to reverse the disadvantage

Table I. Portfolio performance measured against the learning months L. All the other parameters are set as $n = 74$ and $\gamma = 1\%$. First column "$\overline{u(x)}$" shows the ex post mean of the log-utility in percentage, the second shows the turnover in percentage for each portfolio, and the third column "KL Distance" shows the Kullback-Leibler distance from the EP, which is bounded by 4.3

L	$\overline{u(x)}$ (%)			TurnOver (%)			KL Distance		
	SPOP	UP	EP	SPOP	UP	EP	SPOP	UP	EP
3	1.457124	1.487579	1.484813	98.579833	4.569605	4.585246	3.298277	0.0056	0
12	–0.659788	1.473479	1.484813	65.862496	5.047974	4.585246	2.624977	0.019664	0
36	0.729961	1.39445	1.484813	38.581654	6.132685	4.585246	2.24419	0.072564	0
60	0.491328	1.340188	1.484813	26.683572	5.370208	4.585246	1.309914	0.082795	0
120	1.139812	1.352259	1.484813	17.223358	6.128498	4.585246	0.843155	0.178007	0
ALL	1.50841	1.520991	1.484813	4.330022	2.016586	4.585246	0.362633	0.355289	0

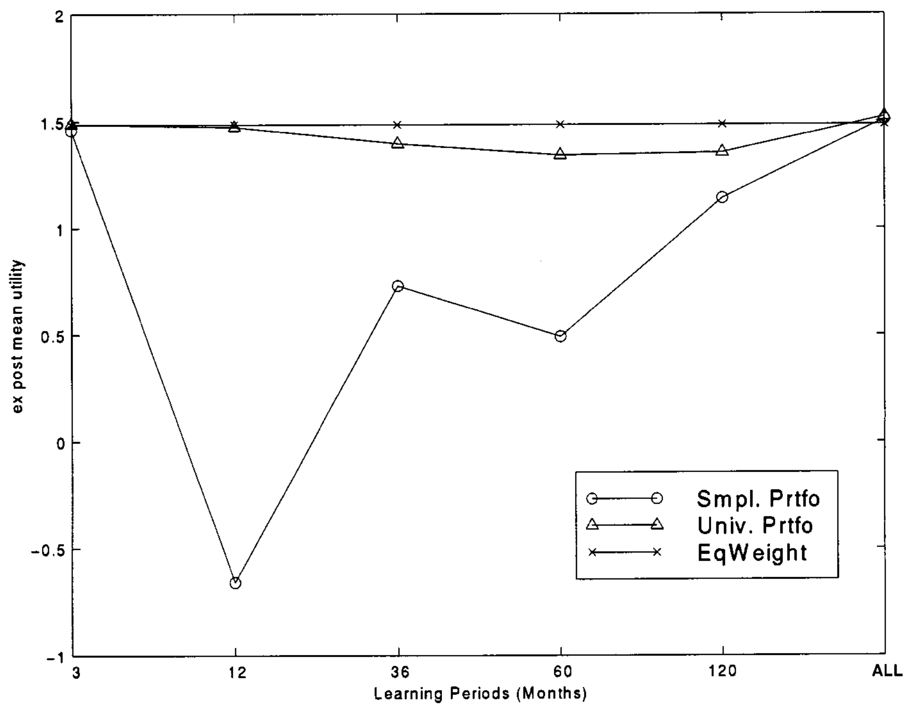

Figure 1. Ex post mean of the log-utility measured against the learning months L. This is a visualization of the first column at Table I. The legend "Smpl. Prtfo" shows the SPOP, "Univ. Prtfo" shows the UP, and "EqWeight" shows the EP.

in transaction costs. And to extract these most successfully, the SPOP with the entire asset universe and three months learning should be used, because the SPOP selects its optimal portfolio in the maximal growth. From the mathematical programming viewpoint, the larger the universe gets, the more likely a bigger objective can be found. But note that this SPOP is no longer a good duplication of the ExLog portfolio and should be regarded as a transitory one. As a counterexample, the U.S. stock market is not efficient in the short term even if the market requires proper costs.

4.3. VARYING TRANSACTION COST RATES γ

The results with $L = ALL$ and $L = 3$ are given in Table IV and Table V, respectively. With $L = ALL$, the order in the ex post utility among the three portfolios basically does not change, according to transaction costs. The differences among portfolios are within 5%. This is because all the portfolios with $L = ALL$ are in the duplicated ExLog portfolios and are almost constant. These then, will not fall victim to transaction costs. Among these, the UP is most robust in costs. The adaptive property of the UP to newly added returns has the advantages of not

Table II. Portfolio performance measured against the security universe n, with all the available learning months. The figures in parentheses in the third column show the maximal KL distance from EP

	$\overline{u(x)}$ (%)			TurnOver (%)			KL Distance		
	SPOP	UP	EP	SPOP	UP	EP	SPOP	UP	EP
27	1.625279	1.607867	1.647735	3.298554	1.972046	3.838582	0.272942	0.330106	0 (3.295837)
74	1.50841	1.520991	1.484813	4.330022	2.016586	4.585246	0.362633	0.355289	0 (4.304065)
295	0.628351	0.936103	1.15648	5.134374	2.176637	4.593715	1.758318	0.528562	0 (5.686975)
749	0.652023	0.950024	1.165783	6.254216	2.249831	5.65667	2.162602	0.578509	0 (6.618739)

Table III. Portfolio performance measured against the security universe n, with three learning months. The figures in parentheses in the third column show the maximal KL distance from EP

	$\overline{u(x;\ )}$ (%)			TurnOver (%)			KL Distance		
	SPOP	UP	EP	SPOP	UP	EP	SPOP	UP	EP
27	0.202003	1.642577	1.647735	98.949365	3.748113	3.838582	2.127756	0.003615	0 (3.295837)
74	1.457124	1.487579	1.484813	98.579833	4.569605	4.585246	3.298277	0.0056	0 (4.304065)
295	–0.855513	1.133832	1.15648	115.947728	4.591277	4.593715	4.738275	0.00532	0 (5.686975)
749	3.09197	1.171303	1.165783	108.998712	5.740708	5.65667	5.875138	0.010179	0 (6.618739)

only attaining the ExLog portfolio but also reducing the redundancy of a perfectly constant portfolio in the presence of costs. By adapting to the winning portfolio as it was in the past, the UP with $L = ALL$ will do well in the next holding period, does not have to change its weights, and keeps the turnover low.

With $L = 3$, the SPOP is quite different from the EP and decreases its ex post utility as cost requirements increase. As pointed out earlier, given no cost requirement, there seems to exist a favorable portfolio to get prominent utility. But such a portfolio is likely to change its weights excessively due to short-term information and be defeated by costs.

5. Conclusion

The results of Ishijima and Shirakawa (1999) in the continuous-time framework hold to some extent in the practical stock market. That is, after long enough learning, the SPOP and the UP converge to the portfolio duplicating the ideal ExLog portfolio. The reason for this conclusion is stated as follows:

1. With all the available learning periods, the ex post utility is obtained within the highest range.
2. Two portfolios steadily learned the portfolios at the same distance from the EP in the KL distance sense.
3. Two portfolios learned to be constant (myopic), as the ExLog portfolio is supposed to be. This can be seen in their turnover which is kept as low as the EP.
4. The theory in the continuous-time framework by Ishijima and Shirakawa (1999) asserts that both the SPOP and the UP are asymptotically optimal for the log-utility investors under incomplete Information 1.

However, this conclusion should not be interpreted to mean log-utility investors could use any learning portfolio to achieve the ExLog portfolio. Concerning the learning periods, the UP is very robust. Moreover, regarding transaction costs, the UP is rarely defeated by them. Even if compared to the EP of a perfectly constant one, the UP is superior. This is because the UP is endowed by its definition with a modestly adaptive property to newly added price-relatives. In this sense, the UP has the possibility to learn the ExLog portfolio most successfully.

Besides the above results verifying the theory of Ishijima and Shirakawa (1999), we have shown some interesting findings regarding the U.S. stock market via this empirical analysis. First, extremely short or long sets of stock prices in the past include worthwhile information for the log-utility investor and affect the ex post portfolio performance. This implies that the permanent components essential for the log-utility investors surely exist, and allows the investors to learn the ExLog portfolio. And the transitory components that affect the holding period performance also exist. These are most successfully extracted by using the SPOP for the larger universe, and have the possibility to provide the prominent utility. The SPOP using these components, however, has to change its weights frequently, and is

Table IV. Portfolio performance measured against the transaction cost rates γ, with all the available learning months. Note that in this table and Table V, the KL distance for both the SPOP and UP is the same at any cost level. This is because transaction costs are not taken into account in searching for the optimal solutions in either portfolio. Just using the optimal portfolio sequence, each portfolio value process with transaction costs defined by (8) is calculated to provide the results

γ (%)	$\overline{u(x;\ )}$ (%)			TurnOver (%)			KL Distance		
	SPOP	UP	EP	SPOP	UP	EP	SPOP	UP	EP
0	1.552051	1.54123	1.531048	0	0	0	0.362633	0.355289	0
1	1.50841	1.520991	1.484813	4.330022	2.016586	4.585246	0.362633	0.355289	0
2	1.464772	1.500813	1.438561	4.296127	2.006244	4.548191	0.362633	0.355289	0

Table V. Portfolio performance measured against the transaction cost rates γ, with three learning months

γ (%)	$\overline{u(x;\,)}$ (%)			TurnOver (%)			KL Distance		
	SPOP	UP	EP	SPOP	UP	EP	SPOP	UP	EP
0	2.674708	1.533655	1.531048	0	0	0	3.298277	0.0056	0
1	1.457124	1.487579	1.484813	98.579833	4.569605	4.585246	3.298277	0.0056	0
2	0.239395	1.441493	1.438561	81.090473	4.532501	4.548191	3.298277	0.0056	0

defeated by transaction costs. Thus, it cannot duplicate the ExLog portfolio. So sets of stock prices in the U.S. stock market are not uniform concerning whether they contain the worthwhile information for the log-utility investors.

Second, the ex post utility obtained by both the SPOP and UP rarely exceeds the one obtained by the EP, under the existence of transaction costs. And the ExLog portfolio is learned to be close to the EP in the KL sense. Taking the above findings into account, the EP can be regarded as the market portfolio for log-utility investors. Recalling that the log-utility investor aims for growth maximization, it becomes clear why fund managers can rarely outperform the EP in the market.

Third, we can say that the U.S. stock market is efficient in the weak form for the log-utility investors, given costs of 1% (Fama, 1970; Fama, 1991). That is, stock returns reflect all the available information which is wothwhile for them, and so any portfolio cannot provide extra ex post utility beyond the market EP. Or viewing these from another aspect, the transaction costs of 1% are the rational requirement for the stock market to prevent any portfolio from extracting transitory information which has close correlation to get the prominent ex post utility.

For the end of this paper, we note that this empirical analysis, as well as the theories of Ishijima and Shirakawa (1999), is valid, not in the framework for the investors having the general concave utility, but in rather narrow framework for the log-utility investors.

Appendix A: Derivation of Portfolio Value After Rebalancing

Here the portfolio value after rebalancing is derived. Let the portfolio and its value at $t-1$ be $\boldsymbol{b}_{t-1}$ and V_{t-1} respectively. The investment on period t is carried out using this portfolio. If the realized price-relative of assets is $\mathbf{X}_t$, then the ratio of ith asset value to the portfolio value is $b^-_{i,t} \triangleq \dfrac{b_{i,t-1}X_{i,t}}{\boldsymbol{b}'_{t-1}\boldsymbol{X}_t}$. If we assume $\Delta_{i,t}$ to be the rebalance amount of asset i, then the ith asset value after rebalancing at time t is

$$V_{i,t} = V_{t-1}b_{i,t-1}X_{i,t} + \Delta_{i,t}\left(1-\gamma\zeta_{i,t}\right), \tag{A.1}$$

where $\zeta_{i,t} \triangleq 1\left(\text{ if } b_{i,t} \geq \kappa_t b^-_{i,t}\right),\ \zeta_{i,t} \triangleq -1\left(\text{ if } b_{i,t} < \kappa_t b^-_{i,t}\right)$, and κ_t is the rebalance adjustment coefficient. And the entire value of the portfolio after rebalancing is

$$V_t = V_{t-1}\sum_{i=1}^{n} b_{i,t-1}X_{i,t} - \gamma\sum_{i=1}^{n}\Delta_{i,t}\zeta_{i,t}\ . \tag{A.2}$$

At time t, the ratio of ith asset value to the entire portfolio value should be $b_{i,t}$, so we obtain

$$b_{i,t} = \frac{V_{i,t}}{V_t} = \frac{V_{t-1}b_{i,t-1}X_{i,t} + \Delta_{i,t}\left(1-\gamma\zeta_{i,t}\right)}{V_{t-1}\sum_{i=1}^{n} b_{i,t-1}X_{i,t} - \gamma\sum_{i=1}^{n}\Delta_{i,t}\zeta_{i,t}}\ .$$

Expanding this equation, we get ρ not depending on i as follows:

$$\begin{aligned}\rho &\overset{\triangle}{=} \sum_{j=1}^{n} b_{j,t-1} X_{j,t} - \gamma \sum_{k=1}^{n} \widetilde{\Delta}_{k,t} \zeta_{k,t} \\ &= \frac{b_{i,t-1}}{b_{i,t}} X_{i,t} + \widehat{\Delta}_{i,t} \left(1 - \gamma \zeta_{i,t}\right) \ , \end{aligned} \tag{A.3}$$

where $\widetilde{\Delta}_{i,t} \overset{\triangle}{=} \dfrac{\Delta_{i,t}}{V_{t-1}}$, $\widehat{\Delta}_{i,t} \overset{\triangle}{=} \dfrac{\widetilde{\Delta}_{i,t}}{b_{i,t}}$. And we solve the equation of ρ about $\widehat{\Delta}_{i,t}$, $\widetilde{\Delta}_{i,t}$ and $\Delta_{i,t}$, then

$$\begin{aligned}\widehat{\Delta}_{i,t} &= \frac{\rho - \frac{b_{i,t-1}}{b_{i,t}} X_{i,t}}{1 - \gamma \zeta_{i,t}} \ , \\ \widetilde{\Delta}_{i,t} &= b_{i,t} \widehat{\Delta}_{i,t} = \frac{b_{i,t}\rho - b_{i,t-1} X_{i,t}}{1 - \gamma \zeta_{i,t}} \ , \\ \Delta_{i,t} &= V_{t-1} \widetilde{\Delta}_{i,t} = V_{t-1} \frac{b_{i,t}\rho - b_{i,t-1} X_{i,t}}{1 - \gamma \zeta_{i,t}} \ . \end{aligned} \tag{A.4}$$

Assigning $\widehat{\Delta}_{i,t}$ into Equation (A.3), then

$$\begin{aligned}\rho &= \sum_{j=1}^{n} b_{j,t-1} X_{j,t} - \gamma \sum_{k=1}^{n} \frac{\left(b_{k,t}\rho - b_{k,t-1} X_{k,t}\right) \zeta_{k,t}}{1 - \gamma \zeta_{k,t}} \\ &= \sum_{j=1}^{n} b_{j,t-1} X_{j,t} + \sum_{k=1}^{n} \left(1 - \frac{1}{1 - \gamma \zeta_{k,t}}\right) \cdot \left(b_{k,t}\rho - b_{k,t-1} X_{k,t}\right) \ . \end{aligned}$$

From above, ρ is

$$\rho = \frac{\sum_{i=1}^{n} \frac{b_{i,t-1} X_{i,t}}{1-\gamma\zeta_{i,t}}}{\sum_{j=1}^{n} \frac{b_{j,t}}{1-\gamma\zeta_{j,t}}} \ .$$

Whilst, assigning $\Delta_{i,t}$ into Equation (A.2), the portfolio value after rebalancing at t becomes

$$\begin{aligned}V_t &= V_{t-1} \left(\sum_{i=1}^{n} b_{i,t-1} X_{i,t} - \gamma \sum_{i=1}^{n} \frac{\left(b_{i,t}\rho - b_{i,t-1} X_{i,t}\right) \zeta_{i,t}}{1 - \gamma \zeta_{i,t}} \right) \\ &= V_{t-1} \left\{ \sum_{i=1}^{n} b_{i,t-1} X_{i,t} + \sum_{i=1}^{n} \left(1 - \frac{1}{1 - \gamma \zeta_{i,t}}\right) \left(b_{i,t}\rho - b_{i,t-1} X_{i,t}\right) \right\} \\ &= V_{t-1} \cdot \rho \\ &= V_{t-1} \cdot \frac{\sum_{i=1}^{n} \frac{b_{i,t-1} X_{i,t}}{1-\gamma\zeta_{i,t}}}{\sum_{j=1}^{n} \frac{b_{j,t}}{1-\gamma\zeta_{j,t}}} \ . \end{aligned}$$

Next, taking care of the sign of the rebalance amount for ith asset, the rebalance adjustment coefficient κ_t defined at Equation (A.1) is determined as

$$\begin{aligned} \Delta_{i,t} &= V_{t-1}\frac{b_{i,t}\rho - b_{i,t-1}X_{i,t}}{1-\gamma\zeta_{i,t}} \gtrless 0 \\ \iff b^-_{i,t} &= \frac{b_{i,t-1}X_{i,t}}{\sum_{j=1}^n b_{j,t-1}X_{j,t}} \gtrless \frac{b_{i,t}\rho}{\sum_{j=1}^n b_{j,t-1}X_{j,t}} \\ \iff \kappa_t &= \frac{\sum_{j=1}^n b_{j,t-1}X_{j,t}}{\rho} \gtrless \frac{b_{i,t}}{b^-_{i,t}} . \end{aligned}$$

Then the rebalance adjustment coefficient κ_t is

$$\kappa_t = \frac{\sum_{i=1}^n \frac{b_{i,t}}{1-\gamma\zeta_{i,t}} \cdot \sum_{j=1}^n b_{j,t-1}X_{j,t}}{\sum_{k=1}^n \frac{b_{k,t-1}X_{k,t}}{1-\gamma\zeta_{k,t}}} .$$

Finally, since turn-over is defined as the ratio of the amount rebalanced, being summed up for all securities in absolute value, relative to the entire portfolio value before rebalancing, then

$$\begin{aligned} \text{Turnover (\%)} &= 100 \cdot \frac{\sum_{i=1}^n |\Delta_{i,t}|}{V_{t-1}\sum_{j=1}^n b_{j,t-1}X_{j,t}} \\ &= 100 \cdot \frac{\sum_{i=1}^n \left| \frac{b_{i,t}\rho - b_{i,t-1}X_{i,t}}{1-\gamma\zeta_{i,t}} \right|}{\sum_{j=1}^n b_{j,t-1}X_{j,t}} \quad \text{(From Equation (A.4))} . \end{aligned}$$

Acknowledgements

We are grateful to the anonymous referees and the Editor-in-Chief, Tsunemasa Shiba, for their valuable comments.

References

Bellman, R. and Kalaba, R. (1957) On the role of dynamic programming in statistical communication theory, *IRE Transactions of the Professional Group on Information Theory*, IT-3:3, pp. 197–203.

Cover, T. M. (1984) An algorithm for maximizing expected log investment return, *IEEE Transactions on Information Theory*, IT-30 (March), pp. 369–373.

Cover, T. M. (1991) Universal portfolios, *Mathematical Finance* **1** (January), 1–29.

Cover, T. M. and Ordentlich, E. (1996) Universal portfolios with side information, *IEEE Transactions on Information Theory*, IT-42 (March), pp. 348–363.

Elton, E. J. and Gruber, M. J. (1995) *Modern Portfolio Theory and Investment Analysis*, 5th edn, John Wiley & Sons.

Fama, E. F. (1970), Efficient Capital Market: A Review of Theory and Empirical Work, *J. Finance*, **25**, 383–417.

Fama, E. F. and French, K. R. (1988), Permanent and Temporary Components of Stock Prices, *J. Polit. Economy* **96**, 246–273.

Fama, E. F. (1991), Efficient Capital Markets: II, *J. Finance* **46**, 1575–1617.

Grauer, R. R. and Hakansson, N. H. (1982) Higher return, lower risk: Historical returns on long-run, actively managed portfolios of stocks, bonds, and bills, 1936–1978, *Financ. Anal. J.* **38** (Mar–Apr), 39–53.

Grauer, R. R. and Hakansson, N. H. (1985) Returns on levered, actively managed long-run portfolios of stocks, bonds, and bills, 1934–1984, *Financ. Anal. J.* **41** (Sep–Oct), 24–43.

Grauer, R. R. and Hakansson, N. H. (1986), A half century of returns on levered and unlevered portfolios of stocks, bonds, and bills, with and without small stocks, *J. Busin.* **59**, 287–318.

Grauer, R. R. and Hakansson, N. H. (1987), Gains from international diversification: 1968–85 returns on portfolios of stocks and bonds, *J. Finance* **42**, 721–739.

Ishijima, H. and Shirakawa, H. (1999) The optimal log-utility asset management under incomplete information, forthcoming in *Asia-Pacific Financial Markets.*

Jamshidian, F. (1992), Asymptotically optimal portfolios, *Mathematical Finance* **2** (April), 131–150.

Konno, H. (1995) *Financial Engineering: Mean-Variance Model and its Extensions* (in Japanese), Nikkagiren.

Mossin, J. (1968) Optimal multiperiod portfolio policies, *J. Busin.* **41** (April), 215–229.

Poterba, J. M. and Summers, L. H. (1988) Mean reversion in stock prices: Evidence and implications, *J. Financ. Econom.* **22**, 27–59.

Shirakawa, H. and Ishijima, H. (1997) Stochastic analysis of continuous time universal portfolio, Working Paper, Tokyo Institute of Technology.