Asia-Pacific Financial Markets **7:** 145–154, 2000.

The Optimal Log-Utility Asset Management under Incomplete Information

HIROSHI ISHIJIMA [1] and HIROSHI SHIRAKAWA [2]

[1] *Faculty of Policy Management, Keio University, 5332, Endoh, Fujisawa 252-8520, Japan, e-mail: hiroshi_ishijima@hi-ho.ne.jp*
[2] *Center for Research in Advanced Financial Technology, Tokyo Institute of Technology, 2-12-1, Oh-Okayama, Meguro-ku, Tokyo 152-8552, Japan, e-mail: shirakawa@craft.titech.ac.jp*

Abstract. In this paper, we propose a theory for deriving the optimal portfolio that assures the log-utility investors of maximizing their expected utility. Restricting investors' information at defined levels, we propose the sample path-wise optimal portfolio (SPOP), which is consistent with the *back-test* framework used in actual investment. It is proven that, at any *finite* terminal time, this SPOP is asymptotically optimal among all the portfolios which are predictable under investors' incomplete information. The optimality is guaranteed by the continuous Bayesian updating formula. Finally, we discuss an algorithm for searching the SPOP, based on asset prices at discrete time intervals.

Key words: asset management, Bayesian formula, incomplete information, log-utility, SPOP.

1. Introduction

Modern portfolio theories have been developed with models which maximize the expected value of von Neumann–Morgenstern utility. The log-utility we are to treat in this paper can be used for reinvesting investors' wealth, and financial literature has shown its important properties. One of those properties is that the degree of Arrow–Pratt's relative risk aversion is one. The optimal portfolio policy for log-utility investors is constant through their i.i.d. investment horizons [1, 27]. Nevertheless, several authors have paid attention to the problem of how to apply these portfolio theories in the practical market where only incomplete information can be obtained. In the past, there were two approaches to treat this problem. In the first approach, the continuous estimation problem for the asset price process is considered [9, 7, 11, 8, 18, 10, 22–24, 19]. In the analyses by [9, 7, 11, 8, 10, 24], the same continuous Bayesian updating scheme of Liptser–Shiryayev is employed [25]. Then, by using different techniques, the optimal portfolio under incomplete information is derived. The second approach for deriving optimal portfolios is different from the first one in that the estimation of asset distribution and the portfolio optimization are jointly executed. Recently, several authors proposed this scheme, called *universal portfolios*. Given the information of observed asset prices alone, the universal portfolio converges to the portfolio which maximizes the expected log-utility. In this area, Cover pioneered the universal portfolio and got an asymptotic result on a general discrete time series [4]. Cover–Ordentlich extended that result with side information taken into consideration [5]. Jamshidian also showed the same results in a continuous time framework [16]. Shirakawa–Ishijima extended the result into the finite time framework and showed that it provides better expected log-utility compared to the portfolio using the estimated mean and covariance from the past [28].

Our motivation is quite similar to these two approaches, and our treatment resembles the latter. However, our approach to treat this problem is different in that we have built a theoretical framework directly applicable to actual investments. Practically, when investors decide their portfolio positions for the next investment period $t+1$, they carry out a so-called *back-test* from past observations of asset returns. That is, any joint observation of asset returns, in each period from $t-L+1$ to t, is presumed to be uniformly distributed with probability $1/L$ occurring in period $t+1$ [12–15]. Then, investors adopt the portfolio which maximizes the expected value of their utility using the above probability distribution. Our question is whether this portfolio is really the most optimal of all the portfolios which are predictable for the available information set. The objective of this paper is to find the answer to this question, i.e., to show the optimality which assures log-utility investors of maximizing their expected utility at any *finite* terminal time.

The paper is organized as follows. In Section 2, the continuous model of the asset price process is defined. After defining the incomplete information given to investors, we address the expected utility maximization problem, whose expectation is considered in the sequence of probability measure $\mathcal{P}^{(k)}$. In Section 3, we propose a portfolio, called the *SPOP*, which maximizes the sample path-wise value of the constant portfolio. Then in Section 4, the asymptotic optimality of the SPOP at any *finite* terminal time, among all the portfolios which are predictable for the incomplete information, is proved by introducing the optimal portfolio based on the continuous Bayesian updating formula (referred to as the *CBOP*) of Liptser–Shiryayev [25]. Finally, in Section 5, we discuss an algorithm to search for the SPOP.

2. The Model

We consider a security market model with n assets whose prices S_{it} $(i = 1 \ldots, n)$ follow geometric Brownian motion:

$$\begin{aligned} \frac{\mathrm{d}S_{it}}{S_{it}} &= \mu_i \mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij} \mathrm{d}W_{jt} \quad (i = 1, \ldots, n)\,, \\ \text{or} \quad (\mathrm{diag}(\boldsymbol{S}_t))^{-1}\mathrm{d}\boldsymbol{S}_t &= \boldsymbol{\mu}\mathrm{d}t + \Sigma \mathrm{d}\mathbf{W}_t\,, \end{aligned} \tag{1}$$

where $\boldsymbol{S}_0$ is constant, $\mathrm{diag}(\boldsymbol{S}_t)$ is a diagonal matrix whose element is $\boldsymbol{S}_t$, and $\Sigma = (\sigma_{ij})_{1\leqslant i,j\leqslant n}$ is a constant diffusion parameter. Here, $\mathbf{W}_t = (W_{1t}, \ldots, W_{nt})'$

denotes an n-dimensional standard Brownian motion on the filtered probability space $(\Omega_W, \{\mathcal{F}_{W,t}; t \geqslant 0\}, \eta)$, where $\mathcal{F}_{W,t}$ is generated by $\sigma(\{\mathbf{W}_u; 0 \leqslant u \leqslant t\})$. On the other hand, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)'$ is a random vector on $(\Omega_\mu, \mathcal{F}_\mu, \nu^{(k)})$ such that induced measure $\nu^{(k)} \circ \boldsymbol{\mu}^{-1}$ by $\boldsymbol{\mu}$ is a multivariate normal distribution $N(\boldsymbol{m}_0^{(k)}, \boldsymbol{\Gamma}_0^{(k)})$ for each $k \in \mathbf{Z}$. Then, we can construct a sequence of the filtered probability space set $\{(\Omega, \{\mathcal{F}_t; t \geqslant 0\}, \mathcal{P}^{(k)}); k \geqslant 1\}$ such that $\Omega = \Omega_\mu \times \Omega_W$, $\mathcal{F}_t = \mathcal{F}_\mu \otimes \mathcal{F}_{W,t}$ and $\mathcal{P}^{(k)} = \nu^{(k)} \otimes \eta$. Now, we consider the investors with the following class of information which is reasonable for the practical market environment.

INFORMATION 1 *Incomplete information: Investors asymptotically have no prior distribution information on the drift parameter* $\boldsymbol{\mu}$. *That is, we consider the limit of probability measure sequence* $\mathcal{P}^{(k)}$, *such that each prior distribution induced by* $\boldsymbol{\mu}$ *follows* $N(\boldsymbol{m}_0^{(k)}, \boldsymbol{\Gamma}_0^{(k)}) \stackrel{\triangle}{=} N(\boldsymbol{m}_0, k\boldsymbol{\Gamma}_0)$ *for each* $k \geqslant 1$, *where* $\boldsymbol{m}_0$ *and* $\boldsymbol{\Gamma}_0$ *are the given constant vector and covariance matrix. Moreover, they are only provided with the information* $\mathcal{G}_t \subset \mathcal{F}_t$ *generated by a realized asset price process of Equation (1):*

$$\mathcal{G}_t \stackrel{\triangle}{=} \sigma(\boldsymbol{S}_u; 0 \leqslant u \leqslant t) . \tag{2}$$

Remark 1. The information $\mathcal{G}_t$ is enough to derive $\boldsymbol{\Sigma}\boldsymbol{\Sigma}'$ exactly, since the Doob–Meyer decomposition of the quadratic process $(\mathrm{d}\boldsymbol{S}_t/\boldsymbol{S}_t)(\mathrm{d}\boldsymbol{S}_t/\boldsymbol{S}_t)'$ yields the finite process $\boldsymbol{\Sigma}\boldsymbol{\Sigma}'t$.

Remark 2. It is well known that if $\boldsymbol{\mu}$ follows the multivariate normal distribution $N(\boldsymbol{m}, \boldsymbol{\Gamma})$ with the density

$$\phi(\boldsymbol{x}) \stackrel{\triangle}{=} \frac{1}{\left(\sqrt{2\pi}\right)^n |\boldsymbol{\Gamma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{m})' \boldsymbol{\Gamma}^{-1} (\boldsymbol{x} - \boldsymbol{m})\right),$$

its differential entropy $h(\boldsymbol{\mu})$ [Th.9.4.1, 3] is given by

$$h(\boldsymbol{\mu}) \stackrel{\triangle}{=} -\int \phi(\boldsymbol{x}) \log \phi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \frac{1}{2} \log(2\pi e)^n |\boldsymbol{\Gamma}| .$$

Then, we can guarantee that the differential entropy of the prior distribution for $\boldsymbol{\mu}$ is asymptotically infinite, since $1/2 \log(2\pi e)^n |\boldsymbol{\Gamma}_0^{(k)}| = k/2 \log(2\pi e)^n |\boldsymbol{\Gamma}_0| \to \infty$ as $k \to \infty$.

We assume that the investors' utility is expressed as the log-utility function

$$u(x) = \log x \ (x > 0) .$$

Then, investors having the log-utility continuously select the optimal portfolios within all of the $\mathcal{G}_t$-predictable portfolios. In addition, the portfolio selection is made within the following feasible region $\mathbf{D}$:

$$\mathbf{D} = \{\boldsymbol{b} \in \mathbf{R}^n \mid \mathbf{A}\boldsymbol{b} \leqslant \boldsymbol{c}, \quad \boldsymbol{b}'\mathbf{1} = 1, \quad \boldsymbol{b} \geqslant 0\} , \tag{3}$$

where $\mathbf{A} \in \mathbf{R}^{l\times n}$ and $\boldsymbol{c} \in \mathbf{R}^{l}$. The instantaneous return of the portfolio value process is given by

$$\frac{\mathrm{d}V_t(\boldsymbol{b}_\bullet)}{V_t} = \boldsymbol{b}_t'(\mathrm{diag}(\boldsymbol{S}_t))^{-1}\mathrm{d}\boldsymbol{S}_t \,.$$

Without loss of generality, we assume that the investor's initial wealth is $V_0 = 1$. Then, we can easily check that the portfolio value at terminal time T is given by

$$V_T(\boldsymbol{b}_\bullet) = \exp\left[-\frac{1}{2}\int_0^T \boldsymbol{b}_u'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_u\mathrm{d}u + \int_0^T \boldsymbol{b}_u'(\mathrm{diag}(\boldsymbol{S}_u))^{-1}\mathrm{d}\boldsymbol{S}_u\right]. \tag{4}$$

The problem we are to treat in this paper emerges, because, under weak Information 1, we *cannot* observe the realization of $\boldsymbol{\mu}$. The terminal expected log-utility maximization problem we treat is

$$\mathbf{P}_1 \left| \begin{array}{l} \underset{\mathbf{b}_\bullet}{\text{maximize}} \;\; \lim_{k\to\infty} E^{(k)}[u(\,V_T(\boldsymbol{b}_\bullet)\,)] \\ \text{subject to} \;\; \boldsymbol{b}_t \in \mathbf{D}\,, \\ \boldsymbol{b}_t \text{ is } \mathcal{G}_t\text{−predictable process}\,, \end{array}\right.$$

where $E^{(k)}[\cdot]$ is the expectation under the probability measure $\mathcal{P}^{(k)}$ and $\mathbf{1}$ is a vector of ones. In the following discussion, we propose a scheme of how to derive the optimal portfolio for $\mathbf{P}_1$ within the $\mathcal{G}_t$-predictable portfolios.

3. The Asymptotic Optimality of the Sample Path-Wise Optimal Portfolio

In this section, we first sketch the framework of the *back-test* which is frequently employed in the actual investment. Divide $[0, t]$ into N intervals $[t_i, t_{i+1})$ $(0 \leqslant i \leqslant N-1)$ such that $t_i \overset{\triangle}{=} \frac{i}{N}t = i\Delta$, and the price differences at each interval are given by $\Delta\boldsymbol{S}_{t_i} = \boldsymbol{S}_{t_{i+1}} - \boldsymbol{S}_{t_i}$. The back-test framework assumes that the next increment $(\mathrm{diag}(\boldsymbol{S}))^{-1}\Delta\boldsymbol{S}$ during $[t, t+\Delta]$ is uniformly distributed on the historical data under the investor's subjective probability distribution $\mathcal{Q}$;

$$E_Q[(\mathrm{diag}(\boldsymbol{S}))^{-1}\cdot\Delta\boldsymbol{S}|\mathcal{G}_t] \simeq \frac{1}{N}\sum_{i=0}^{N-1}(\mathrm{diag}(\boldsymbol{S}_{t_i})\cdot\Delta\boldsymbol{S}_{t_i})\,.$$

Then, we can easily see that

$$E_{\mathcal{Q}}\left[\log\left(\frac{V_{t+\Delta}(\boldsymbol{b}_\bullet)}{V_t(\boldsymbol{b}_\bullet)}\right)\middle|\mathcal{G}_t\right] \cong \frac{1}{N}\sum_{i=1}^{N}\log\left(1+\boldsymbol{b}_t'\left(\mathrm{diag}\left(\boldsymbol{S}_{t_{i-1}}\right)\right)^{-1}\cdot\Delta\boldsymbol{S}_{t_i}\right). \tag{5}$$

If we maximize the left-hand side of (5) continuously, we can optimize $\mathbf{P}_1$ for the subjective probability measure $\mathcal{Q}$. From Ito's lemma, if we take the limit $N \to \infty$,

$$\sum_{i=1}^{N} \log\left(1 + \boldsymbol{b}_t' \left(\mathrm{diag}\left(\boldsymbol{S}_i\right)\right)^{-1} \cdot \Delta \boldsymbol{S}_i\right) \to -\frac{1}{2}\boldsymbol{b}_t' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}_t t + \\ + \int_0^t \boldsymbol{b}_t' \left(\mathrm{diag}\left(\boldsymbol{S}_u\right)\right)^{-1} \cdot d\boldsymbol{S}_u, \quad \mathcal{P}^{(k)}\text{-}a.s.$$

Then, the back-test framework of the portfolio optimization results in the following parametric problem with respect to t:

$$\mathbf{P}_1(t) \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & V_t(\boldsymbol{b}) = \exp\left[\left(\boldsymbol{b}'\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}\right)t + \boldsymbol{b}'\Sigma\mathbf{W}_t\right] \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\,. \end{array}\right.$$

Here, note that $\mathbf{P}_1(t)$ is equivalent to the following problem,

$$\mathbf{P}_2(t) \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{minimize}} & \frac{1}{2}\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b} - \boldsymbol{b}'\tilde{\boldsymbol{\mu}}_t \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\,, \end{array}\right.$$

where

$$\begin{aligned} \tilde{\boldsymbol{\mu}}_t &\stackrel{\triangle}{=} \boldsymbol{\mu} + 1/t\,\boldsymbol{\Sigma}\boldsymbol{W}_t \\ &= \frac{1}{t}\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1}\mathrm{d}\boldsymbol{S}_u\,. \end{aligned} \tag{6}$$

This problem is well defined under Information 1. Hereafter, we will call the optimal solution $\boldsymbol{b}_t^*$ of $\mathbf{P}_2(t)$ based on the back-test framework the *sample path-wise optimal portfolio (SPOP)*. Also, note that the SPOP is $\mathcal{G}_t$-predictable, since it seeks the constant portfolio which maximizes the sample path-wise portfolio value at time t according to one sample path $\{\boldsymbol{S}_u; 0 \leqslant u \leqslant t\}$.

Next, we shall prove the asymptotic optimality of the SPOP among all the $\mathcal{G}_t$-predictable portfolios, using the continuous Bayesian updating scheme. The portfolio selection problem under incomplete information, which is quite similar to our model, has been considered by several authors [9, 7, 11, 8, 10, 24]. Let us consider the portfolio learning scheme of Gennotte [11], based on the continuous Bayesian updating formula of Liptser–Shiryayev [25]. We suppose that the investors only have Information 1. Then, as we described in Remark 1, they know the diffusion parameter $\boldsymbol{\Sigma}\boldsymbol{\Sigma}'$ exactly, but do not know the $\mathcal{F}_\mu$-measurable drift parameter $\boldsymbol{\mu}$. Hereafter, as noted in Remark 2, we will consider the sequence of probability measures $\{\mathcal{P}^{(k)}, k \in \mathbf{Z}\}$ on the filtered probability space

$(\{\mathcal{F}_t; t \geqslant 0\}, \Omega)$, where the prior distribution for $\boldsymbol{\mu}$ follows $\nu^{(k)}$. Utilizing the information $\mathcal{G}_t$, the investors estimate $\boldsymbol{\mu}$ as follows:

$$\boldsymbol{m}_t^{(k)} \triangleq E^{(k)}\left[\boldsymbol{\mu} \mid \mathcal{G}_t\right],$$
$$\boldsymbol{\Gamma}_t^{(k)} \triangleq E^{(k)}\left[\left(\boldsymbol{\mu} - \boldsymbol{m}_t^{(k)}\right)\left(\boldsymbol{\mu} - \boldsymbol{m}_t^{(k)}\right)' \middle| \mathcal{G}_t\right],$$

where $E^{(k)}[\cdot]$ denotes the expectation under $\mathcal{P}^{(k)}$, $\boldsymbol{m}_t^{(k)}$ is the estimation for $\boldsymbol{\mu}$, and $\boldsymbol{\Gamma}_t^{(k)}$ is its estimation error. Using infinitesimal observations $\mathrm{d}\boldsymbol{S}_t$, we can improve the estimation by the continuous Bayesian updating formula of [25] to obtain the improvement $\mathrm{d}\boldsymbol{m}_t^{(k)}$. By assumption, we have $E^{(k)}[|\boldsymbol{\mu}|^4] = E^{(k)}\left[|\boldsymbol{\mu}|^4 \middle| \mathcal{G}_0\right] < \infty$. According to Theorem 12.8 in [25], the estimation of $\boldsymbol{\mu}$ conditioned on $\mathcal{G}_t$ is given by

$$\begin{aligned}\boldsymbol{m}_t^{(k)} &= \left[\mathbf{I} + \boldsymbol{\Gamma}_0^{(k)}\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1} t\right]^{-1} \cdot \left[\boldsymbol{m}_0^{(k)} + \boldsymbol{\Gamma}_0^{(k)}\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1} \textstyle\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1}\,\mathrm{d}\boldsymbol{S}_u\right] \\ &= \left[\frac{1}{k}\boldsymbol{\Gamma}_0^{-1} + \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1} t\right]^{-1} \cdot \left[\frac{1}{k}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{m}_0 + \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1}\,\mathrm{d}\boldsymbol{S}_u\right],\end{aligned} \tag{7}$$

where $\mathbf{I}$ is an identity matrix. Since we aim to analyze the infinite differential entropy case, we should take the limit $k \to \infty$. Then, the estimator $\boldsymbol{m}_t^{(k)}$ converges to

$$\lim_{k\to\infty} \boldsymbol{m}_t^{(k)} = \frac{1}{t}\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1}\,\mathrm{d}\boldsymbol{S}_u = \tilde{\boldsymbol{\mu}}_t, \quad P\text{-}a.s., \tag{8}$$

where $\tilde{\boldsymbol{\mu}}_t$ is given by Equation (6). Next transform Equation (1) into entirely observable s.d.e. :

$$(\mathrm{diag}(\boldsymbol{S}_t))^{-1}\,\mathrm{d}\boldsymbol{S}_t = \boldsymbol{m}_t^{(k)}\mathrm{d}t + \boldsymbol{\Sigma}\mathrm{d}\tilde{\mathbf{W}}_t^{(k)}, \tag{9}$$

where

$$\tilde{\mathbf{W}}_t^{(k)} \triangleq \boldsymbol{\Sigma}^{-1}\left[\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1}\,\mathrm{d}\boldsymbol{S}_u - \int_0^t \boldsymbol{m}_u^{(k)}\mathrm{d}u\right] = \boldsymbol{\Sigma}^{-1}\int_0^t (\boldsymbol{\mu} - \boldsymbol{m}_u^{(k)})\mathrm{d}u + \mathbf{W}_t.$$

Then, $\tilde{\mathbf{W}}_t^{(k)}$ is $\mathcal{G}_t$-measurable, and

$$E^{(k)}\left[\tilde{\mathbf{W}}_t^{(k)} - \tilde{\mathbf{W}}_s^{(k)} \middle| \mathcal{G}_s\right] = E^{(k)}\left[\boldsymbol{\Sigma}^{-1}\int_s^t (\boldsymbol{\mu} - \boldsymbol{m}_u^{(k)})\mathrm{d}u \middle| \mathcal{G}_s\right] = 0.$$

Furthermore, $\left\langle \tilde{\mathbf{W}}^{(k)} \right\rangle_t = \langle \mathbf{W} \rangle_t = t$. Hence, $\tilde{\mathbf{W}}_t^{(k)}$ is $\mathcal{G}_t$ standard Brownian motion owing to Lévy's theorem under the probability measure $\mathcal{P}^{(k)}$. Then, the portfolio value at terminal time T, using the $\mathcal{G}_t$-predictable drift $\boldsymbol{m}_t^{(k)}$ and portfolios $\boldsymbol{b}_\bullet$, is given by

$$V_T(\boldsymbol{b}_\bullet) = \exp\left[-\tfrac{1}{2}\textstyle\int_0^T \boldsymbol{b}_u'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_u\mathrm{d}u + \int_0^T \boldsymbol{b}_u'\left(\boldsymbol{m}_u^{(k)}\mathrm{d}u + \boldsymbol{\Sigma}\mathrm{d}\tilde{\mathbf{W}}_u^{(k)}\right)\right], \quad k \geqslant 1. \tag{10}$$

The problem $\mathbf{P}_1$ without convex constraints for the portfolio weights is explicitly solved by Lakner [23, 24]. In Lakner's approach, filtering techniques are introduced to treat $\mathbf{P}_1$, i.e., the optional projection of the Radon–Nikodym derivative to $\mathcal{G}_t$ is considered and an explicit solution of $\mathbf{P}_1$ is derived. Also, in the complete market with complete information $\mathcal{F}_t$, explicit solutions without convex constraints are obtained in [17] and with convex constraints in [6]. Recently, under incomplete information, explicit solutions were also obtained in [19], using both martingale and partial differential equation methodologies. Here, we emphasize that the optimal solution of $\mathbf{P}_1$ can be also obtained by the SPOP under Information 1. The advantage of the SPOP is that the optimal portfolio under incomplete information can be obtained by just maximizing the sample path-wise value. Then, it can be directly applicable in actual investments. Let $\boldsymbol{b}_t^{*(k)}$ be the optimal portfolio under $\mathcal{P}^{(k)}$ for the problem $\mathbf{P}_1$, which is based on the continuous Bayesian formula (referred to as the CBOP). In the following theorem, we prove the optimality of CBOP $\boldsymbol{b}_t^{*(k)}$ and SPOP $\boldsymbol{b}_t^*$ among all the $\mathcal{G}_t$-predictable portfolios .

THEOREM 1 (Optimality of $\boldsymbol{b}_t^*$) *Under Information 1 and with the observable s.d.e. (9), the asymptotic optimal portfolio is given by the SPOP, $\boldsymbol{b}_\bullet^*$, for any finite terminal time T. That is*

$$(\{\forall \mathcal{G}_t\text{-predictable } \boldsymbol{b}_t \in \mathbf{D};\ 0 \leqslant t \leqslant T\})(E^{(k)}[u(V_T(\boldsymbol{b}_\bullet))] \leqslant E^{(k)}[u(V_T(\boldsymbol{b}_\bullet^{*(k)}))]),$$

and

$$\lim_{k\to\infty} E^{(k)}[u(V_T(\boldsymbol{b}_\bullet^*))] = \lim_{k\to\infty} E^{(k)}[u(V_T(\boldsymbol{b}_\bullet^{*(k)}))] .$$

Proof. Using (10), we have

$$\begin{aligned} J &\stackrel{\triangle}{=} E^{(k)}[u(V_T(\boldsymbol{b}_\bullet))] \\ &= E^{(k)}\left[-\frac{1}{2}\int_0^T \boldsymbol{b}_u' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}_u \mathrm{d}u + \int_0^T \boldsymbol{b}_u' \left(\boldsymbol{m}_u^{(k)}\mathrm{d}u + \boldsymbol{\Sigma}\mathrm{d}\tilde{\mathbf{W}}_u^{(k)}\right)\right] . \end{aligned}$$

Here, we note that $\boldsymbol{b}_u \in \mathbf{D}$ and $\mathbf{D}$ is compact. Hence, $\int_0^T \boldsymbol{b}_u' \boldsymbol{\Sigma}\mathrm{d}\tilde{\mathbf{W}}_u^{(k)}$ is martingale under $\left(\Omega, \{\mathcal{G}_t\}_{t\geq 0}, \mathcal{P}^{(k)}\right)$, which guarantees $E^{(k)}[\int_0^T \boldsymbol{b}_u'\boldsymbol{\Sigma}\mathrm{d}\tilde{\mathbf{W}}_u^{(k)}] = 0$. Then,

$$J = \int_0^T E^{(k)}\left[-\frac{1}{2}\boldsymbol{b}_u' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}_u + \boldsymbol{b}_u' \boldsymbol{m}_u^{(k)}\right]\mathrm{d}u .$$

Thus, to maximize J, one needs only to find the optimal portfolio $\boldsymbol{b}_u^{*(k)}$ that maximizes $-1/2\boldsymbol{b}_u'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_u + \boldsymbol{b}_u'\boldsymbol{m}_u^{(k)}$ for each u. Then, $\boldsymbol{b}_t^{*(k)}$ is the optimal solution for $\mathbf{P}_2(t)$, where $\boldsymbol{m}_t^{(k)}$ is substituted for $\tilde{\boldsymbol{\mu}}_t$. Furthermore, since $\boldsymbol{m}_t^{(k)} \to \tilde{\boldsymbol{\mu}}_t$ from (8), $\boldsymbol{b}_t^{*(k)}$ converges to $\boldsymbol{b}_t^*$ with probability one. Therefore,

$$\lim_{k\to\infty} \left|E^{(k)}[u(\ V_T(\boldsymbol{b}_\bullet^*)\ )] - E^{(k)}[u(\ V_T(\boldsymbol{b}_\bullet^{*(k)})\ )]\right| = 0 .$$

□

This result asserts that the CBOP, $\boldsymbol{b}_t^{*(k)}$, maximizes the expected log-utility under $\mathcal{P}^{(k)}$, and $\boldsymbol{b}_t^{*(k)}$ converges to the SPOP, $\boldsymbol{b}_t^*$, asymptotically.

4. An Algorithm for the SPOP at Discrete Time Intervals

Since the continuous model enables us to guarantee the asymptotic optimality of the SPOP, we then focus on the algorithm for searching the SPOP in discrete time. The difficulty here is that since asset prices in practical markets are observed only once in a discrete time interval, the maximal expected concave utility portfolio cannot be directly obtained from these observations, and we cannot make assumptions on asset prices other than as to their nonnegativity. To address these limitations, we utilize the results in the continuous model and derive the algorithm for the SPOP.

Again, we assume that n assets exist in the market, and that the price-relative vector of assets is $\boldsymbol{X}_t = \left(X_{1,t}, \ldots, X_{n,t}\right)'$. It is assumed that the prices of each security $S_{i,t}$ are observed on the market in discrete time $t = 0, 1, \ldots$, and so its price relatives $X_{i,t} = \frac{S_{i,t}}{S_{i,t-1}}$.

According to the continuous theory discussed in an earlier section, if the investor has the log-utility, the optimal portfolio is the SPOP at any finite terminal time. Hence, at any time $t > 0$, the log-utility investor's optimal policy is stated as

$$\mathbf{P}_3 \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & V_t(\boldsymbol{b}) = \prod_{u=1}^{t}(1 + \boldsymbol{b}'(\mathbf{X}_u - \mathbf{1})) = \prod_{u=1}^{t} \boldsymbol{b}'\boldsymbol{X}_u \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\,. \end{array} \right.$$

This is equivalent to the following problem:

$$\mathbf{P}_3' \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & \frac{1}{t}\sum_{u=1}^{t} \log \boldsymbol{b}'\boldsymbol{X}_u \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\,. \end{array} \right.$$

Hereafter, we abbreviate the operation $\frac{1}{t}\sum_{u=1}^{t}$ to E.

The algorithm for searching for the optimal solution of $\mathbf{P}_3'$, without the constraint $\mathbf{A}\boldsymbol{b} \leqslant \boldsymbol{c}$, is given by Cover [2]. His algorithm is simply written as follows: at some feasible point $\boldsymbol{b}^{(k)}$, the next new point $\boldsymbol{b}^{(k+1)}$ is given by $\boldsymbol{b}^{(k+1)} = \operatorname{diag}(\boldsymbol{b}^{(k)})E\left[\frac{\boldsymbol{X}}{\boldsymbol{b}^{(k)'}\boldsymbol{X}}\right]$. For this algorithm, the monotonic improvement of the objective and its convergence to the optimal solution satisfying the KKT condition [20, 21] are guaranteed. Noting that $E\left[\frac{\boldsymbol{X}}{\boldsymbol{b}^{(k)'}\boldsymbol{X}}\right]$ is the gradient vector of the objective function in $\mathbf{P}_3'$, this algorithm can be viewed as a special type of *gradient projection method*. To treat $\mathbf{P}_3'$ with the constraint $\mathbf{A}\boldsymbol{b} \leqslant \boldsymbol{c}$, we can employ the gradient projection method if we are to extend Cover's algorithm using the steepest ascent direction, i.e., the gradient itself. In the gradient projection method, the gradient is projected onto $M \stackrel{\triangle}{=} \{\boldsymbol{x} \in \mathbf{R}^q \mid \mathbf{A}_q\boldsymbol{x} = 0\}$, where $\mathbf{A}_q \in \mathbf{R}^{q\times n}$ is composed of rows of active constraints in $\mathbf{A}$. Then, we are able to make new points iteratively by making use of the projected gradient until the new point satisfies the KKT condition. For further details about this method, including its monotonic improvement of the objective and its convergence, refer to [26].

5. Conclusion

In this paper, we have shown that the log-utility maximization portfolio under incomplete information can be obtained with the SPOP, either in theory or by using the algorithm. First, we showed that the SPOP means the back-test framework of continuous portfolio selection under incomplete information. We also verified the asymptotic optimality of the SPOP by proposing the CBOP, under the prior distribution for $\boldsymbol{\mu}$ with infinite differential entropy. The question regarding the validity of this SPOP in the general security market has been left for future theoretical analysis.

Acknowledgements

We are grateful to the referees and Tsunemasa Shiba, Editor-in-Chief, who offered discerning comments on earlier drafts, greatly improving them in the process. We are also grateful to IBJ-DL Financial Technology for their financial support.

References

1. Bellman, R. and Kalaba, R. (1957) On the role of dynamic programming in statistical communication theory, *IRE Trans. Professional Group on Information Theory* **IT-3:3**, 197–203.
2. Cover, T. M. (1984) An algorithm for maximizing expected log investment return, *IEEE Trans. Inform. Theory* **IT-30**, 369–373.
3. Cover, T. M. and Thomas, J. A. (1990) *Elements of Information Theory*, John Wiley & Sons.
4. Cover, T. M. (1991) Universal portfolios, *Math. Finance* **1**, 1–29.
5. Cover, T. M. and Ordentlich, E. (1996) Universal portfolios with side information, *IEEE Trans. Inform. Theory* **IT-42**, 348–363.
6. Cvitanić, J. and Karatzas, I. (1992) Convex duality in constrained portfolio optimization, *Ann. Appl. Prob.* **2** (4), 767–818.
7. Detemple, J. B. (1986) Asset pricing in a production economy with incomplete information, *J. Finance* **41** (2), 383–392.
8. Detemple, J. B. (1991) Further results on asset pricing with incomplete information, *J. Econom. Dynam. Control* **15** (3), 425–453.
9. Dothan, M. U. and Feldman, D. (1986) Equilibrium interest rates and multiperiod bonds in a partially observable economy, *J. Finance* **41** (2), 369–382.
10. Feldman, D. (1992) Logarithmic preferences, myopic decisions, and incomplete information, *J. Financ. Quantitat. Anal.* **27** (4), 619–630.
11. Gennotte, G. (1986) Optimal portfolio choice under incomplete information, *J. Finance* **41** (3), 733–746.
12. Grauer, R. R. and Hakansson, N. H. (1982) Higher return, lower risk: Historical returns on long-run, actively managed portfolios of stocks, bonds, and bills, 1936–1978, *Financ. Anal. J.* **38** (Mar–Apr), 39–53.

13. Grauer, R. R. and Hakansson, N. H. (1985) Returns on levered, actively managed long-run portfolios of stocks, bonds, and bills, 1934–1984, *Financ. Anal. J.* **41** (Sep–Oct), 24–43.
14. Grauer, R. R. and Hakansson, N. H. (1986) A half century of returns on levered and unlevered portfolios of stocks, bonds, and bills, with and without small stocks, *J. Busin.* **59**, 287–318.
15. Grauer, R. R. and Hakansson, N. H. (1987) Gains from international diversification: 1968–85 returns on portfolios of stocks and bonds, *J. Finance* **42**, 721–739.
16. Jamshidian, F. (1992) Asymptotically optimal portfolios, *Math. Finance* **2**, 131–150.
17. Karatzas, I., Lehoczky, J. P., and Shreve, S. E. (1987) Optimal portfolio and consumption decisions for a 'small investor' on a finite horizon, *SIAM J. Control Optim.* **25** (6), 1557–1586.
18. Karatzas, I. and Xue, X. (1991) A note on utility maximization under partial observations, *Math. Finance* **1** (2), 57–70.
19. Karatzas, I. and Zhao, X. (1998) Bayesian adaptive portfolio optimization, preprint.
20. Karush, W. (1939) Minima of functions of several variables with inequalities as side constraints, MSc dissertation, Department of Mathematics, University of Chicago.
21. Kuhn, H. W. and Tucker, A. W. (1950) Nonlinear programming. In N. C. Metropolis (ed.), *Proc. Second Berkeley Symp. Mathematical Statistics and Probability*, University of California Press.
22. Kuwana, Y. (1995) Certainty equivalence and logarithmic utilities in consumption/investment problems, *Math. Finance* **5** (4), 297–309.
23. Lakner, P. (1995) Utility maximization with partial information, *Stoch. Proces. Appl.* **56**, 247–273.
24. Lakner, P. (1998) Optimal trading strategy for an investor: the case of partial information, *Stoch. Proces. Appl.* **76**, 77–97.
25. Liptser, R. S. and Shiryayev, A. N. (1978) *Statistics of Random Processes II: Applications*, Springer-Verlag.
26. Luenberger, D. G. (1984) *Linear and Nonlinear Programming*, 2nd edn, Addison-Wesley Publ.
27. Mossin, J. (1968) Optimal multiperiod portfolio policies, *J. Busin.* **41**, 215–229.
28. Shirakawa, H. and Ishijima, H. (1997) Stochastic analysis of continuous time universal portfolio, Working Paper, Tokyo Institute of Technology.