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The Regime Switching Portfolios

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Abstract In this paper we develop a portfolio selection theory under regime switching means and volatilities. We use log mean-variance as the portfolio selection criteria and, as a result, the theory is made substantially easier to implement than other existing theories. Moreover, the estimated regimes are easy to interpret as one of the regimes corresponds to the business cycle turning points. Finally, we conduct an asset allocation simulation and obtain reasonable results by introducing an idea of switching volatility targets.

Keywords Markov switching model · Continuous-and discrete-time regime switching · Log mean-variance · Portfolio selection · EM algorithm

1 Introduction

The theory and application of portfolio selection have been a central topic of research in finance; especially recent developments in an intertemporal setting, or 'strategic asset allocation', are striking (e.g., Campbell and Viceira 2002). However, one of the strict

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assumptions made in most of literatures is that the security returns are independently and identically distributed. The most common assumption is that the security returns follow an identical normal distribution, although many researchers have shown that this does not match the actual data and numerous kinds of models have been proposed.

One way of departing from the simple normality assumption is to consider a mixture of normal distributions. Hamilton (1989) proposed a model called Markov switching model, where the mean and the variance of a stochastic process depend on an unobservable, Markov state variable. While Elliott and van der Hoek (1997) study a one-period portfolio selection problem, we model the security returns using this model and consider an intertemporal portfolio selection problem under the regime switching returns and volatilities. With the same motivation, Ang and Bekaert (2002) numerically treat the problem in a discrete-time setting. In contrast to their approach, we first develop a general, continuous-time portfolio selection model under continuous-time regime switching. We employ the familiar method using the Hamilton–Jacobi–Bellman, (HJB), equation to solve the continuous-time portfolio selection problem. This can be considered a version of Merton's problem (Merton 1969, 1971) and enables us to show the optimal portfolio under regime switching.

The development up to this point may be enough from a theoretical aspect. We proceed further, however, to consider a continuous-time log mean-variance portfolio selection under discrete-time regime switching. Our prime motivation in this paper is the practical modeling of portfolio selection under regime switching; this scheme is the one that matches our purpose.

There is a reason for adopting the log mean-variance model as a continuous-time portfolio selection scheme. The log mean-variance model is equivalent to maximizing the mean growth-rate of the portfolio value, or wealth, with its variance held low. Its fascinating properties have been advocated since Kelly (1956), Breiman (1961), Thorp (1971), and many others (e.g., Cover and Thomas 1991). Of course, the log mean maximization, or the expected log-utility maximization, will not maximize the other types of utility functions. However, Luenberger (1993) has shown that the log mean-variance model has a valid long-run motivation, in the sense of *tail preference*. Moreover, Konno et al. (1993) have developed a fast algorithm for calculating the efficient frontier with respect to the log mean-variance tradeoff. This algorithm can be easily implemented, even when we incorporate constraints which would be incurred in practice.

The reason for assuming the regime switching to occur at deterministic and discrete-time intervals is that there are established theoretical and empirical results on regime switching models in the time series literature. For example, Krolzig (1997) has successfully implemented this model to detect the switches between recessions and booms in the business cycle of several countries. Schaller and van Norden (1997), and Maheu and McCurdy (2000), also used this model to identify bull and bear markets.

An empirical analysis is conducted using the theory described above. First we estimate the model parameters using the bond and stock return data. The estimation yields interesting information which is easy to interpret and consistent with former findings. Next, an asset allocation simulation is conducted. We consider an optimal mix of bonds and stocks and, using an idea of 'switching volatility targets,' we achieve better performance than stocks alone. Although the performance of our portfolio is not as high as we expected, it is good enough to outperform the stock index.

To summarize, this paper's contribution is both theoretical and empirical: Firstly, we provide a practical theory of portfolio selection under regime switching returns and volatilities. Secondly, the estimation result of the Markov switching model gives new insights regarding the security market. Thirdly, an introduction of switching volatility targets significantly increased the performance of the portfolio.

This paper is organized as follows. Section two develops the theory based on log-mean variance criteria, under the continuous- and discrete-time regime switching means and volatility. Section three introduces the Markov switching model and the estimation procedures are discussed in detail. Section four gives the estimation result and the interpretation of the estimated states. Several interesting interpretation of the results are presented. Section five conducts an asset allocation simulation and, finally, section six concludes the paper.

2 Theory of Regime Switcing Portfolios

2.1 Portfolio Selection Under Continuous-Time Regime Switching

We consider a market in which n risky assets are traded. The investment horizon is $[0, T]$. We assume that the price process of assets follows the stochastic differential equation (s.d.e.):

$$(\mathrm{diag}\boldsymbol{S}_t)^{-1}\, d\boldsymbol{S}_t = \boldsymbol{\mu}_t dt + \Sigma_t d\boldsymbol{W}_t, \tag{1}$$

where $\mathrm{diag}\boldsymbol{S}_t$ is a diagonal matrix whose elements are $\boldsymbol{S}_t$. Here $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_{1t}, \dots, W_{nt})'\,; t \geq 0\}$ denotes an n-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, the superscript $'$ denotes the transpose, and $\mathcal{F}_t^W \stackrel{\triangle}{=} \sigma(\boldsymbol{W}_u; 0 \leq u \leq t)$.

It is assumed that there exists K economic regimes which *continuously* switch the drift parameter $\boldsymbol{\mu}_t$ and the diffusion parameter Σ_t. The economic regime at time t is denoted by a K-dimensional column vector Y_t, which has a realization in $\{e_1, \dots, e_k, \dots, e_K\}$, where e_k is a vector whose kth element is 1 and otherwise 0. Here we suppose that the drift parameter $\boldsymbol{\mu} = \{\boldsymbol{\mu}_t; t \geq 0\}$ and the diffusion parameter $\Sigma = \{\Sigma_t; t \geq 0\}$ take K realizations depending on the economic regime Y.

To be more precise, first define an indicator function

$$I_t(k) \stackrel{\triangle}{=} \langle Y_t, e_k\rangle = \begin{cases} 1 & (\text{if } Y_t = e_k) \\ 0 & (\text{otherwise}), \end{cases} \tag{2}$$

where the operator $\langle , \rangle$ denotes an inner product. We assume there are K different vectors $\{\boldsymbol{\mu}(1), \dots, \boldsymbol{\mu}(K)\}$ and matrices $\{\Sigma(1), \dots, \Sigma(K)\}$, such that

$$\boldsymbol{\mu}_t = \sum_{k=1}^{K} I_t(k)\boldsymbol{\mu}(k) \stackrel{\triangle}{=} \boldsymbol{\mu}(Y_t), \tag{3}$$

$$\Sigma_t = \sum_{k=1}^{K} I_t(k)\Sigma(k) \overset{\triangle}{=} \Sigma(Y_t). \tag{4}$$

Writing $\Lambda(k) \overset{\triangle}{=} \Sigma(k)\Sigma(k)'$, we abbreviate $\Sigma_t\Sigma_t'$ to be

$$\Lambda_t = \Sigma_t\Sigma_t' = \sum_{k=1}^{K} I_t(k)\Sigma(k)\Sigma(k)' \overset{\triangle}{=} \Lambda(Y_t). \tag{5}$$

We describe the economic regime Y_t as a continuous-time Markov switching process, whose characteristics are the following. $Y = \{Y_t; t \geq 0\}$ is also defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, presumed to be independent of the Brownian motion $\boldsymbol{W}$, and we write $\mathcal{F}_t^Y \overset{\triangle}{=} \sigma(Y_u; 0 \leq u \leq t)$. Define $\boldsymbol{\xi}_t \overset{\triangle}{=} E[Y_t]$. Then we have the backward Kolmogorov equation for $\boldsymbol{\xi}_t$:

$$\frac{d\boldsymbol{\xi}_t}{dt} = \mathbf{Q}\boldsymbol{\xi}_t, \tag{6}$$

where $\mathbf{Q} = (q_{kl})_{1\leq k,l\leq K}$ is defined as a Q-matrix:

$$q_{kl} = \begin{cases} \lim_{h\to 0} \frac{P(Y_{t+h}=e_l|Y_t=e_k)}{h} & (\text{if } k \neq l) \\ \lim_{h\to 0} \frac{1-P(Y_{t+h}=e_l|Y_t=e_l)}{h} & (\text{if } k = l) \end{cases}. \tag{7}$$

Hence $q_{ll} = -\sum_{k=1,k\neq l}^{K} q_{kl}$. By the result shown by the appendix B of Elliott et al. (1995), the regime Y_t has the dynamics

$$dY_t = \mathbf{Q}Y_t dt + dM_t, \tag{8}$$

where $M = \{M_t; t \geq 0\}$ is a martingale with respect to $\mathcal{F}^Y$ and assumed to be independent of $\boldsymbol{W}$.

We suppose that the investor's utility is given by a twice continuously differentiable, strictly concave function $u(x)(x > 0)$ with $u'(0) = \infty$ and $u'(\infty) = 0$. The investor continuously selects portfolios so as to maximize the expected utility from the terminal wealth. The portfolio is characterized by a weight $\boldsymbol{b}_t \in \mathbb{R}^n$ whose ith element $b_{i,t}(i = 1, \dots, n)$ is the ratio of wealth invested in the ith asset to the entire wealth. In this subsection, it is assumed that a risk-free asset is also traded in the market with a price process given by

$$dr_t = \mu_0 r_t dt. \tag{9}$$

Here μ_0 is a constant. By the budget constraint, the ratio invested in the risk-free asset is $1 - \boldsymbol{b}_t'\mathbf{1}$, where $\mathbf{1}$ is an n-dimensional vector of ones. Then the instantaneous return of the portfolio value (wealth) process is given by

$$\begin{aligned}\frac{dV_t}{V_t} &= \boldsymbol{b}_t' \left(\mathrm{diag}\boldsymbol{S}_t\right)^{-1} d\boldsymbol{S}_t + \left(1 - \boldsymbol{b}_t'\mathbf{1}\right) \frac{dr_t}{r_t} \\ &= \left\{\boldsymbol{b}_t' \left(\boldsymbol{\mu}_t - \mu_0 \mathbf{1}\right) + \mu_0\right\} dt + \boldsymbol{b}_t' \Sigma_t d\boldsymbol{W}_t. \end{aligned} \tag{10}$$

In the above expression, the portfolio selection is constructed without constraints. Then, with $\mathcal{F}_t^{W,Y} \stackrel{\triangle}{=} \sigma(\boldsymbol{W}_u, Y_u; 0 \leq u \leq t)$, we state the investor's objective as the following problem:

$$\mathbf{P}_1 \left| \begin{array}{ll} \max\limits_{\boldsymbol{b}_\bullet} & E\left[u\left(V_T(\boldsymbol{b}_\bullet)\right)\right] \\ \text{subject to} & \boldsymbol{b}_t \text{ is } \mathcal{F}_t^{W,Y} - \text{predictable.} \end{array} \right. \tag{11}$$

To solve Problem $\mathbf{P}_1$, define the value function at t as:

$$J\left(V_t, Y_t, t\right) \stackrel{\triangle}{=} \max_{\{\boldsymbol{b}_u; t \leq u \leq T\}} E\left[u\left(V_T(\boldsymbol{b}_\bullet)\right) \middle| \mathcal{F}_t^{W,Y}\right]. \tag{12}$$

Recall the economic regime $Y_t \in \mathcal{F}_t^{W,Y}$ has a realization in $\{e_1, \ldots, e_K\}$. Then with the expression

$$\boldsymbol{J}\left(V_t, t\right) = \left(J\left(V_t, e_1, t\right), \ldots, J\left(V_t, e_K, t\right)\right), \tag{13}$$

the value function can be written as

$$J\left(V_t, Y_t, t\right) = \left\langle \boldsymbol{J}\left(V_t, t\right),\ Y_t \right\rangle. \tag{14}$$

The HJB equation is:

$$0 = \max_{\boldsymbol{b}_t}\ \phi_t\left(\boldsymbol{b}_t, Y_t\right) \tag{15}$$

where $\phi_t\left(\boldsymbol{b}_t, Y_t\right)$ is given by

$$\phi_t\left(\boldsymbol{b}_t, Y_t\right) = J_V V_t \left\{\boldsymbol{b}_t' \left(\boldsymbol{\mu}_t - \mu_0 \mathbf{1}\right) + \mu_0\right\} + J_t + \frac{1}{2} J_{VV} V_t^2 \boldsymbol{b}_t' \Lambda_t \boldsymbol{b}_t + \boldsymbol{J}\left(V_t, t\right) \mathbf{Q} Y_t. \tag{16}$$

When $Y_t = e_k$ $(k = 1, \ldots, K)$, we write the expressions of (3), (5) and (16) as

$$\begin{aligned} \boldsymbol{\mu}_t &= \boldsymbol{\mu}(k), \\ \Lambda_t &= \Lambda(k), \\ \boldsymbol{\phi}_t(\boldsymbol{b}_t) &= \left(\phi_t(\boldsymbol{b}_t, e_1), \ldots, \phi_t(\boldsymbol{b}_t, e_K)\right). \end{aligned}$$

Then the HJB of (16) reduces to K partial differential equations:

$$\begin{aligned} 0 &= \max_{\boldsymbol{b}_t} \ \phi_t(\boldsymbol{b}_t, e_k) \\ &= \max_{\boldsymbol{b}_t} \ J_V V_t \left\{\boldsymbol{b}_t' \left(\boldsymbol{\mu}(k) - \mu_0 \mathbf{1}\right) + \mu_0\right\} + J_t + \frac{1}{2} J_{VV} V_t^2 \boldsymbol{b}_t' \Lambda(k) \boldsymbol{b}_t + \boldsymbol{J}\left(V_t, t\right) \mathbf{Q} e_k, \\ &\quad (k = 1, \ldots, K). \end{aligned} \tag{17}$$

Using first order conditions, we obtain the optimal portfolio $\boldsymbol{b}_t^*$ according to the realization of the economic regime Y_t at time t:

$$\boldsymbol{b}_t^* = \boldsymbol{b}_t^*(Y_t) = -\frac{J_V}{J_{VV} V_t} \sum_{k=1}^{K} I_t(k) \Lambda^{-1}(k) \left(\boldsymbol{\mu}(k) - \mu_0 \mathbf{1}\right), \quad (Y_t = e_1, \ldots, e_K) \tag{18}$$

We call $\boldsymbol{b}_t^*(Y_t)$ the *regime switching portfolio*. There is no hedging portfolio since the opportunity set is constant as long as the regime does not change. Hence the optimal decision of an investor under our regime switching setting is to prepare K distinct portfolios that are optimal in each regime and switch among them according to the economic state at each point of time.

For general utility functions, it is difficult to solve the p.d.e. (16) when $\boldsymbol{b}_t^*(Y_t)$ is substituted for $\boldsymbol{b}_t$. If, however, we specify the utility to be the log-type, on which we will construct a portfolio selection model in the next subsection, we are able to derive the optimal portfolio in a closed-form.

Theorem 1 (The regime switching log portfolio) *Under the model setting which admits the asset price process of* (1) *to switch its parameters,* (3) *and* (4)*, according to the economic regimes which has the dynamics of* (8)*, the optimal portfolio for the log investor is given by*

$$\boldsymbol{b}_t^{\log *}(Y_t) = \sum_{k=1}^{K} I_t(k) \Lambda^{-1}(k) \left(\boldsymbol{\mu}(k) - \mu_0 \mathbf{1}\right). \tag{19}$$

Proof As in Merton (1971) and Cox et al. (1985), the value function of (12) has the form of

$$J(V_t, Y_t, t) = f(t) \log V_t + g(Y_t, t). \tag{20}$$

$f(t)$ and $g(Y_t, t)$ are given as solutions to the p.d.e. of (16) in which $\boldsymbol{b}_t^*(Y_t)$ of (18) is substituted for $\boldsymbol{b}_t$ and (20) for the value function. The boundary condition is $f(T) = 1$ and $g(Y_T, T) = 0$. Then we have $\frac{J_V}{J_{VV} V_t} = -1$ in (18) to complete the proof. □

2.2 Log Mean-Variance Portfolio Selection Under Discrete-Time Regime Switching

Again, we consider a market in which n assets are traded and their price processes follow the s.d.e. (1) as before. The investment horizon is again $[0, T]$. It is assumed that, however, the economic regime switches occur *discretely*, namely, at deterministic discrete times: $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_N = T$. We call the time interval $(t_{i-1}, t_i]$ the *period* t_i $(i = 1, \ldots, N)$. Then we describe the economic regime as a discrete-time Markov switching model. Again the process of the economic regimes is denoted by $Y \stackrel{\triangle}{=} \{Y_{t_i};\ i = 1, 2, \ldots\}$, and Y_{t_i} has state space $\{e_1, \ldots, e_k, \ldots, e_K\}$. As in the foregoing continuous-time analysis, we assume that its transition probability matrix is time-homogeneous and denoted by

$$\mathbf{P} = (p_{kl})_{1 \le k,l \le K} = \left(\Pr\left(Y_{t_{i+1}} = e_k \middle| Y_{t_i} = e_l\right)\right)_{1 \le k,l \le K}. \tag{21}$$

That is, the (row k, column l)-element of $\mathbf{P}$, p_{kl}, represents the probability of switching from state e_l to e_k. Writing $\mathcal{F}^Y_{t_i} \stackrel{\triangle}{=} \sigma\left(Y_{t_j};\ j = 1, \ldots, i\right)$, our Markov model is describe as:

$$\begin{aligned} &E\left[Y_{t_{i+1}} \middle| Y_{t_i} = e_l\right] = E\left[Y_{t_{i+1}} \middle| Y_{t_i} = e_l, \mathcal{F}^Y_{t_{i-1}}\right] = \mathbf{P}e_l, \\ &\text{or } E\left[Y_{t_{i+1}} \middle| Y_{t_i}\right] = \mathbf{P}Y_{t_i}. \end{aligned} \tag{22}$$

Then the economic regime has the dynamics

$$Y_{t_{i+1}} = \mathbf{P}Y_{t_i} + M_{t_{i+1}}, \tag{23}$$

where $M = \{M_{t_i};\ i = 1, 2, \ldots\}$ is an $\mathcal{F}^Y$-martingale increment. Note that the economic regime Y_{t_i} is assumed to be an $\mathcal{F}^Y_{t_i}$-measurable random vector. Again we define the $\mathcal{F}^Y_{t_i}$-measurable indicator function as follows:

$$I_{t_i}(k) \stackrel{\triangle}{=} \langle Y_{t_i}, e_k \rangle = \begin{cases} 1 & (\text{if } Y_{t_i} = e_k), \\ 0 & (\text{otherwise}). \end{cases} \tag{24}$$

For any time t in the period t_i $(i = 1, \ldots, N)$, it is assumed that there are K different vectors $\{\boldsymbol{\mu}(1), \ldots, \boldsymbol{\mu}(K)\}$ and matrices $\{\Sigma(1), \ldots, \Sigma(K)\}$, such that

$$\boldsymbol{\mu}_t = \sum_{k=1}^{K} I_{t_i}(k)\boldsymbol{\mu}(k) \stackrel{\triangle}{=} \boldsymbol{\mu}(Y_{t_i}), \tag{25}$$

$$\boldsymbol{\Sigma}_t = \sum_{k=1}^{K} I_{t_i}(k)\Sigma(k) \stackrel{\triangle}{=} \Sigma(Y_{t_i}). \tag{26}$$

Writing $\Lambda(k) \stackrel{\triangle}{=} \Sigma(k)\Sigma(k)'(k = 1, \ldots, K)$, we abbreviate $\Sigma_t \Sigma_t'$ as

$$\Lambda_t = \Sigma_t \Sigma_t' = \sum_{k=1}^{K} I_{t_i}(k)\Sigma(k)\Sigma(k)' \stackrel{\triangle}{=} \Lambda(Y_{t_i}). \tag{27}$$

Then we rewrite the asset price processes (1) for the time t in the period t_i ($i = 1, \ldots, N$), conditioned on the $\mathcal{F}_{t_i}^Y$-measurable economic regime Y_{t_i}, as:

$$(\mathrm{diag}\boldsymbol{S}_t)^{-1}\, d\boldsymbol{S}_t \Big| Y_{t_i} = \boldsymbol{\mu}(Y_{t_i})dt + \Sigma(Y_{t_i})d\boldsymbol{W}_t. \tag{28}$$

The investor constructs his portfolio for the period t_i according to the regime Y_{t_i}. We denote by $\boldsymbol{b}_t$ the portfolio weights, the ratio of the amount invested in the ith asset to the entire portfolio value at time t. It is supposed that the portfolio selection is made within the following feasible region:[1]

$$\mathbf{D} \stackrel{\triangle}{=} \left\{\boldsymbol{b} \in \mathbb{R}^n \,\Big|\, \boldsymbol{b}'\mathbf{1} = 1, \boldsymbol{b} \geq \mathbf{0}\right\}. \tag{29}$$

In the foregoing section, we discussed the portfolio selection based on the expected utility. For reasons given at the head of this section, we adopt the *log mean-variance model* for the portfolio selection.

Regime Switching Log Mean-Variance Model

In the regime switching log mean-variance model, the investor's objective is to maximize the expected logarithm of the terminal portfolio value, or the expected growth rate of the portfolio value, subject to the regime switching, (or state-dependent) log-variance over the entire investment horizon.

We remark that although our model flexibly varies the target log-variance level conditioned on the economic regime, it does *NOT* change the objective of the expected logarithm from the terminal portfolio value. To state the portfolio choice problem for the log mean-variance model, we shall derive the expectation and variance of the terminal log portfolio value in our setting. For the period t_i, the instantaneous return of the portfolio value, conditioned on the economic regime Y_{t_i}, is given by:

$$\left.\frac{dV_t}{V_t}\right| Y_{t_i} = \boldsymbol{b}_t'\boldsymbol{\mu}(Y_{t_i})dt + \boldsymbol{b}_t'\Sigma(Y_{t_i})d\boldsymbol{W}_t. \tag{30}$$

Applying the Ito's rule, we obtain:

$$d\log V_t \mid Y_{t_i} = \mu_P\left(\boldsymbol{b}_t;\ Y_{t_i}\right)dt + \boldsymbol{b}_t'\Sigma(Y_{t_i})d\boldsymbol{W}_t, \tag{31}$$

[1] In addition to (29), one can impose the inequality $\mathbf{A}\boldsymbol{b} \leq \boldsymbol{c}$, for the practical purpose. Here $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{c} \in \mathbb{R}^m$. For example, the sponsor may claim that the total weight of specified assets should be less than c. Even if we address this constraint, the discussion in the rest of this subsection is essentially the same.

where

$$\mu_P\left(\boldsymbol{b}_t; Y_{t_i}\right) \stackrel{\triangle}{=} \boldsymbol{b}_t' \boldsymbol{\mu}(Y_{t_i}) - \frac{1}{2}\boldsymbol{b}_t' \Lambda(Y_{t_i})\boldsymbol{b}_t. \tag{32}$$

Assuming $V_0 = 1$, the conditional expectation from the terminal log portfolio value is:

$$\begin{aligned} E\left[\log V_T \middle| \mathcal{F}_T^Y\right] &= E\left[\sum_{i=1}^{N}\left\{\int_{t_{i-1}}^{t_i} d\log V_t \middle| Y_{t_i}\right\}\middle| \mathcal{F}_T^Y\right] \\ &= \sum_{i=1}^{N} E\left[E\left[\int_{t_{i-1}}^{t_i} \mu_P\left(\boldsymbol{b}_t;\ Y_{t_i}\right)dt + \int_{t_{i-1}}^{t_i} \boldsymbol{b}_t'\Sigma(Y_{t_i})d\boldsymbol{W}_t \middle| \mathcal{F}_{t_{i-1}}^W, \mathcal{F}_T^Y\right]\middle| \mathcal{F}_T^Y\right] \\ &= \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} \mu_P\left(\boldsymbol{b}_t; Y_{t_i}\right)dt. \end{aligned} \tag{33}$$

The last equation is obtained since $\int_{t_{i-1}}^{t_i} \boldsymbol{b}_t'\Sigma(Y_{t_i})d\boldsymbol{W}_t$ is an $\mathcal{F}_{t_i}^W \times \mathcal{F}_T^Y$-martingale. From above, to maximize the conditional expectation from the terminal log portfolio value under the constraint $\mathbf{D}$ of (29), it is enough to maximize $\mu_P\left(\boldsymbol{b}_t; Y_{t_i}\right)$ for each period t_i $(i = 1, \ldots, N)$. Then the optimal portfolio at any time in period t_i is constant, conditioned on the economic regime Y_{t_i}. Specifically, if the maximization is implemented *without* the constraint $\mathbf{D}$ as in the previous subsection, the optimal portfolio $\boldsymbol{b}_t$ at any time t in the period t_i is obtained as

$$\max_{\boldsymbol{b}_t} \mu_P\left(\boldsymbol{b}_t; Y_{t_i}\right).$$

From (32), the optimal portfolio is given as

$$\begin{aligned} \boldsymbol{b}_t^{\log *}(Y_{t_i}) = \boldsymbol{b}_{t_i}^{\log *}(Y_{t_i}) &= \Lambda(Y_{t_i})^{-1}\boldsymbol{\mu}(Y_{t_i}) \\ &= \sum_{k=1}^{K} I_{t_i}(k)\Lambda(k)^{-1}\boldsymbol{\mu}(k) \ \left(\forall t \in (t_{i-1}, t_i]\right). \end{aligned} \tag{34}$$

Again, the optimal log portfolio is the regime switching portfolio of (19) in which the expected return $\boldsymbol{\mu}(k)$ is substituted for the expected excess return $\boldsymbol{\mu}(k) - \mu_0\mathbf{1}$. We then obtain the unconditional expectation from the terminal log portfolio value:

$$E[\log V_T] = E\left[E\left[\log V_T \middle| \mathcal{F}_T^Y\right]\right] = \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} \mu_P(\boldsymbol{b}_t)dt, \tag{35}$$

where

$$\mu_P(\boldsymbol{b}_t) \triangleq \sum_{k=1}^{K} E[I_{t_i}(k)]\mu_P(\boldsymbol{b}_t; e_k). \tag{36}$$

Evaluating the conditional log variance of the portfolio value over the whole investment horizon, we have from (33) and the Ito isometry,

$$\begin{aligned} V\left[\log V_T \mid \mathcal{F}_T^Y\right] &= E\left[\left(\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} \boldsymbol{b}_t'\Sigma(Y_{t_i})d\boldsymbol{W}_t\right)^2 \middle| \mathcal{F}_T^Y\right] \\ &= \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} \sigma_P^2\left(\boldsymbol{b}_t; Y_{t_i}\right) dt. \end{aligned} \tag{37}$$

Here

$$\sigma_P^2\left(\boldsymbol{b}_t;\; Y_{t_i}\right) \triangleq \boldsymbol{b}_t'\Lambda(Y_{t_i})\boldsymbol{b}_t. \tag{38}$$

To state the problem for the regime switching log mean-variance model, we constrain the above log variance of the portfolio value over the entire investment horizon, in addition to the constraint $\mathbf{D}$ of (29). Since the log variance of each period may vary according to the $\mathcal{F}_{t_i}^Y$-measurable economic regime Y_{t_i}, we constrain $\sigma_P^2(\boldsymbol{b}_t; Y_{t_i})$ in (37) to be $\sum_{k=1}^{K} I_{t_i}(k)\bar{\sigma}^2(k)$ in each period t_i $(i = 1, \ldots, N)$, where $\bar{\sigma}^2(k)$, $(k = 1, \ldots, K)$ are the K target log variances set by the investor a priori. Hence,

$$V\left[\log V_T \mid \mathcal{F}_T^Y\right] = \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} \sigma_P^2\left(\boldsymbol{b}_t; Y_{t_i}\right) dt = \sum_{i=1}^{N}\left\{(t_i - t_{i-1})\sum_{k=1}^{K} I_{t_i}(k)\bar{\sigma}^2(k)\right\}.$$

Taking a second expectation with respect to Y for both sides to obtain the unconditional log variance of the portfolio value:

$$\begin{aligned} V[\log V_T] &= E\left[V\left[\log V_T \mid \mathcal{F}_T^Y\right]\right] \\ &= \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} \sigma_P^2(\boldsymbol{b}_t)dt = \sum_{i=1}^{N}\sum_{k=1}^{K}\left\{E\left[I_{t_i}(k)\right]\bar{\sigma}^2(k)(t_i - t_{i-1})\right\}, \end{aligned}$$

where

$$\sigma_P^2(\boldsymbol{b}_t) \triangleq \sum_{k=1}^{K} E\left[I_{t_i}(k)\right]\sigma_P^2(\boldsymbol{b}_t; e_k). \tag{39}$$

Finally we are able to state the portfolio selection problem for the log mean-variance model for each period t_i:

$$\mathbf{P}_{t_i} \left| \begin{array}{ll} \max\limits_{\boldsymbol{b}_{t_i}} & \mu_P(\boldsymbol{b}_{t_i}) \\ \text{subject to } \sigma_P^2(\boldsymbol{b}_{t_i}) = & \sum_{k=1}^K E[I_{t_i}(k)]\bar{\sigma}^2(k), \\ & \boldsymbol{b}_{t_i} \in \mathbf{D}. \end{array} \right. \tag{40}$$

3 Markov Switching Model

In the following sections we implement the theory developed above. This section briefly discusses the basic setup of the empirical model and its estimation procedure for Markov switching models. Markov switching models are advantageous since they are flexible enough to describe the characteristics of actual security returns. Parameter estimation is conducted using the maximum-likelihood method, but due to the hidden state variables, a special recursive filter proposed by Hamilton (1989, 1994) is required.[2] There are two methods for maximizing the likelihood function, a numerical optimization and an EM algorithm. We adopt the latter method in the parameter estimation and this will be explained.

3.1 Basic Setup

Assuming $t_i - t_{i-1} = h$, where $h \triangleq \dfrac{T}{N}$, we discretize the s.d.e. (28) as follows:

$$\boldsymbol{R}_{t_i} \big| Y_{t_i} \triangleq (\text{diag}(\boldsymbol{S}_{t_{i-1}}))^{-1}(\boldsymbol{S}_{t_i} - \boldsymbol{S}_{t_{i-1}}) \Big| Y_{t_i} = \boldsymbol{\mu}(Y_{t_i})h + \Sigma(Y_{t_i})\sqrt{h}\boldsymbol{\epsilon}_{t_i}, \tag{41}$$

$$\text{or} \quad \boldsymbol{R}_{t_i} \big| Y_{t_i} \sim N\left(\boldsymbol{\mu}(Y_{t_i})h,\ \Lambda(Y_{t_i})h\right) \quad (i = 1, \ldots, N) \tag{42}$$

where $\epsilon_{t_i} \sim N(\mathbf{0}, \mathbf{I})$ and we write $\mathcal{R}_{t_i} \triangleq \left(\boldsymbol{R}_{t_1}, \ldots, \boldsymbol{R}_{t_i}\right)$. In other words, we assume that the returns follow a multivariate normal distribution with its means and variances switching depending on a countable number of states. The state is assumed to switch according to an unobservable Markov process with a transition probability matrix $\mathbf{P}$, whose klth element is the probablity of switching from the state e_l to e_k, $i.e. p_{kl} = P(Y_{t_i} = e_k | Y_{t_{i-1}} = e_l)$. Hamilton (1989, 1994) has proposed a recursive filter, by which the probability of being at each state can be obtained. This filter allows us to distinguish among different states and utilized in our portfolio selection model.[3]

The advantage for using this kind of model is its flexibility in realizing various distributions. Timmermann (2000) has shown that the model is able to generate distributions with excess kurtosis and negative skewness that are evident in many security return

[2] Elliott (1994, 1995) also studies the adaptive filters for hidden Markov chains and related processes in detail.

[3] The model can be extended so that the transition probabilities themselves depend on some exogenous variables. See Diebold et al. (1994) and Filardo (1994) for such extensions.

data. Although a mixture of independent normal distributions is also able to achieve this, the independence assumption is somewhat contrary to intuition. For example, considering two states representing good and bad states of an economy, it is not natural to assume the state changes independently. Some kind of dependence assumption is more realistic, and in fact, Markov switching model is successfully implemented to determine switches between recessions and booms in the real sector of the economy (Krolzig 1997), and bull and bear markets in financial markets (Schaller and van Norden 1997; Maheu and McCurdy 2000).

Although there are other methods for estimating the state probability, Markov switching models are advantageous by the following two reasons. First, Layton and Katsuura (2001) compare MS, probit and logit models, which are popular methods for estimating the state probability, and conclude that Markov switching model performs the best for diagnosing the business cycle transitions. Although there is no such study conducted in financial markets, this result should be highly relevant. Second, while the Markov switching model can be estimated without arbitrariness, logit and probit models require a 'true' state as a dependent variable to estimate the coefficients. This kind of arbitrariness is precluded in Markov switching models.

3.2 Estimation Procedures

The estimation of Markov switching model is conducted using maximum-likelihood, but due to the hidden state variables, it is more complicated than standard settings. Before discussing the estimation procedures we define a few variables below.

First, we define a vector of conditional density $\boldsymbol{\eta}$:

$$\boldsymbol{\eta}_{t_i} \overset{\triangle}{=} \left(f(\boldsymbol{R}_{t_i}|Y_{t_i}=e_1;\boldsymbol{\theta})\dots f(\boldsymbol{R}_{t_i}|Y_{t_i}=e_k;\boldsymbol{\theta})\dots f(\boldsymbol{R}_{t_i}|Y_{t_i}=e_K;\boldsymbol{\theta})\right)'. \quad (43)$$

We assume that the returns are conditionally normally distributed. That is, for $\boldsymbol{R}_{t_i} \in \mathbb{R}^n$, each conditional density function can be written as:

$$f(\boldsymbol{R}_{t_i}|Y_{t_i}=e_k;\boldsymbol{\theta}) = (2\pi)^{-\frac{n}{2}}|\boldsymbol{\Lambda}(k)|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}(\boldsymbol{R}_{t_i}-\boldsymbol{\mu}(k))'\boldsymbol{\Lambda}(k)^{-1}(\boldsymbol{R}_{t_i}-\boldsymbol{\mu}(k))\right\}, \quad (44)$$

for $k=1,\dots,K$. The parameter vector $\boldsymbol{\theta}$ contains the conditional means, variances, covariances and transition probability parameters at each state.

Next, we define the vector of estimation for the state $\boldsymbol{\xi}_{t_i|t_j}$:

$$\boldsymbol{\xi}_{t_i|t_j} \overset{\triangle}{=} \left(P(Y_{t_i}=e_1|\mathcal{R}_{t_j})\dots P(Y_{t_i}=e_k|\mathcal{R}_{t_j})\dots P(Y_{t_i}=e_K|\mathcal{R}_{t_j})\right)'. \quad (45)$$

Since $\mathcal{R}_{t_j}$ represents the history of the return $\boldsymbol{R}$ upto the time t_j, $\boldsymbol{\xi}_{t_i|t_j}$ contains the state probabilities at time t_i based on the information up to time t_j. This is called a *filter* if $t_i = t_j$, a *forecast* if $t_i > t_j$ and a *smoother* if $t_i < t_j$. With an initial value $\boldsymbol{\xi}_{0|0}$, the filter and the one-step-ahead forecast can be calculated by the following iteration scheme:

$$\boldsymbol{\xi}_{t_i|t_i} = \frac{\boldsymbol{\xi}_{t_i|t_{i-1}} \odot \boldsymbol{\eta}_{t_i}}{\mathbf{1}'(\boldsymbol{\xi}_{t_i|t_{i-1}} \odot \boldsymbol{\eta}_{t_i})}, \tag{46}$$

$$\boldsymbol{\xi}_{t_{i+1}|t_i} = \boldsymbol{P} \cdot \boldsymbol{\xi}_{t_i|t_i}, \tag{47}$$

where $\odot$ denotes the element-by-element multiplication of vectors. The initial value $\boldsymbol{\xi}_{0|0}$ can be another vector of unknown variables to be estimated by maximum-likelihood. The denominator of (46) actually calculates the likelihood.[4] Therefore, with T numbers of data observation, the log-likelihood function to maximize can be written as:

$$l(\boldsymbol{\theta}) = \sum_{i=1}^{N} \log \mathbf{1}'(\boldsymbol{\xi}_{t_i|t_{i-1}} \odot \boldsymbol{\eta}_{t_i}). \tag{48}$$

There are mainly two methods for maximizing this function: a numerical optimization and an EM algorithm. Both procedures have pros and cons as shown by Mizrach and Watkins (1999). They compare the two methods and find that while the calculation speed of the former is faster, the latter is more robust with more ill shaped likelihood functions. Since we treat a fairly complicated model with two variables and two to three states, we adopt the EM algorithm.

The general EM algorithm was proposed by Dempster et al. (1977) and the application to the Markov switching model was studied by Hamilton (1990). This is fairly easily implemented using the filter and the smoother described above. According to Kim (1993), the smoother is defined as:

$$\boldsymbol{\xi}_{t_i|T} = \boldsymbol{\xi}_{t_i|t_i} \odot \left\{\boldsymbol{P}' \left[\boldsymbol{\xi}_{t_{i+1}|T} (\div) \boldsymbol{\xi}_{t_{i+1}|t_i}\right]\right\}, \tag{49}$$

where $(\div)$ denotes the element-by-element division of vectors. The initial value $\boldsymbol{\xi}_{T|T}$ is given by the last step of the recursive calculation (46), (47) and then this is solved backward to obtain $\boldsymbol{\xi}_{t_i|T}, i = 1, \ldots, N$.

This algorithm consists of the following two steps:

1. (Expectation Step): Run the filter (46), (47) and the smoother (49) to obtain $\Pr\left(Y_{t_i} = e_k | \mathcal{R}_T; \boldsymbol{\theta}\right)$, for $k = 1, \ldots, K,\ i = 1, \ldots, N$.
2. (Maximization Step): Update the parameters according to the following rule (Hamilton 1990; Diebold et al. 1994):

$$\boldsymbol{\mu}(k)^{(j+1)} = \frac{\sum_{i=1}^{N} P(Y_{t_i} = e_k|\mathcal{R}_T; \boldsymbol{\theta}^{(j)}) \boldsymbol{R}_{t_i}}{\sum_{i=1}^{N} P(Y_{t_i} = e_k|\mathcal{R}_T; \boldsymbol{\theta}^{(j)})}, \quad \text{for } k = 1, \ldots, K, \tag{50}$$

$$\boldsymbol{\Lambda}(k)^{(j+1)} = \frac{\sum_{i=1}^{N} P(Y_{t_i} = e_k|\mathcal{R}_T; \boldsymbol{\theta}^{(j)})(\boldsymbol{R}_{t_i} - \boldsymbol{\mu}(k)^{(j+1)})(\boldsymbol{R}_{t_i} - \boldsymbol{\mu}(k)^{(j+1)})'}{\sum_{i=1}^{N} P(Y_{t_i} = e_k|\mathcal{R}_T; \boldsymbol{\theta}^{(j)})}, \quad \text{for } k = 1, \ldots, K, \tag{51}$$

4 See Hamilton (1994) for details.

$$p_{kl}^{(j+1)} = \frac{\sum_{i=2}^{N} \Pr\left(Y_{t_i} = e_k, Y_{t_{i-1}} = e_l | \mathcal{R}_T; \boldsymbol{\theta}^{(j)}\right)}{\sum_{i=2}^{N} \Pr\left(Y_{t_{i-1}} = e_l | \mathcal{R}_T; \boldsymbol{\theta}^{(j)}\right)},$$
$$\text{for } k, l = 1, \ldots, K, \tag{52}$$

where the superscript (j) shows that the corresponding parameter is obtained in the jth recursive calculation. Setting the initial value $\boldsymbol{\theta}^{(0)}$, the two steps are repeated until a predetermined convergence criterion is achieved. Although the EM algorithm is known to be robust to the choices of initial values, several different values should be tested in order to avoid local maxima. This algorithm is implemented by programming in C++, which significantly speeds up the estimation.

4 Application to the Japanese Security Markets

4.1 Data

In the following analysis, we consider two assets: the Japanese bond and stock. For bond data, we use the Nomura BPI index, which is constructed using government and corporate bonds issued in Japan with ratings above A, excluding convertible and mortgage bonds and asset backed securities. This index was set as 100 at the end of December 1983. For the stock data, we use TOPIX, which is a value-weighted index of the first section in the Tokyo Stock Exchange. The index was set at 100 in January 4, 1968. The data used here is monthly. It starts in April 1972 and ends in July 2001, giving to 352 monthly data points. We make use of the maximum number of data available since to obtain reliable estimates the MS model requires far more data than ordinary single state models.[5]

4.2 Full Sample Estimation

Tables 1, 2, 3, Figs. 1 and 2 show the estimation results for two and three state models using the full sample explained above. We first analyze the two-state model. From the estimated parameters in Table 1, we can interpret that the first state corresponds to the 'bad' market condition and the second state to the 'normal' one. In the second state, bonds and stocks are both performing fairly well, with moderate level of volatility, while both perform poorly with a higher level of volatility in the first state. The stock market in the first state is remarkably turbulent with negative expected returns and a volatility that is three times higher than the one in second state.

The transition probabilities in Table 2 show that the second state is highly persistent compared to the first. We can calculate the expected duration for each state as $1/(1 - p_{ii})$, where the p_{ii} shows the probability of staying state i. According to this, the former is expected to last for about three months, while the latter is expected for more than a year. This shows that, although the first state corresponds to a bad market

[5] See Psaradakis and Sola (1998) for details.

Table 1 Estimated parameters for the two- and three-state model

	Two state model	Three state model
State 1		
Bond return (%)	0.314 (0.248)	0.538 (0.106)
Stock return (%)	−0.461 (1.037)	1.117 (0.267)
Bond variance	3.640 (0.567)	0.836 (0.120)
Stock variance	57.436 (10.230)	6.624 (1.092)
Bond-stock covariance	1.553 (1.811)	0.233 (0.230)
Expected duration	3.300	9.259
State 2		
Bond return (%)	0.639 (0.054)	0.718 (0.101)
Stock return (%)	0.695 (0.294)	0.108 (0.531)
Bond variance	0.581 (0.057)	0.457 (0.082)
Stock variance	18.326 (1.890)	31.189 (3.797)
Bond-stock covariance	0.439 (0.236)	0.731 (0.427)
Expected duration	12.658	7.874
State 3		
Bond return (%)		0.281 (0.288)
Stock return (%)		−0.364 (1.209)
Bond variance		4.147 (0.678)
Stock variance		65.096 (12.419)
Bond-stock covariance		1.994 (2.328)
Expected duration		3.534
Log-likelihood	−1561.50	−1541.71

Standard errors are reported in the parenthesis. The expected durations for each state is calculated as $1/(1 - p_{ii})$, where the p_{ii} shows the probability of staying at the state i. All parameters are monthly

Table 2 Transition probability matrix for the two-state model

	From State 1	From State 2
To State 1	0.697 (0.089)	0.079 (0.029)
To State 2	0.303 (0.109)	0.921 (0.051)

Standard errors are reported in the parenthesis

Table 3 Transition probability matrix for the three-state model

	From State 1	From State 2	From State 3
To State 1	0.892 (0.065)	0.044 (0.065)	0.170 (0.092)
To State 2	0.082 (0.076)	0.873 (0.075)	0.114 (0.105)
To State 3	0.026 (0.029)	0.082 (0.045)	0.717 (0.037)

Standard errors are reported in the parenthesis

condition, it does not last as long as the second one. In other words, the market stays calm for about a year and then a short turbulent period occurs. Since the expansions and recessions last for 33.1 and 17.4 months on average, (see Table 4), the security market seems to have a shorter cycle which is closely related to the real sector of the economy.

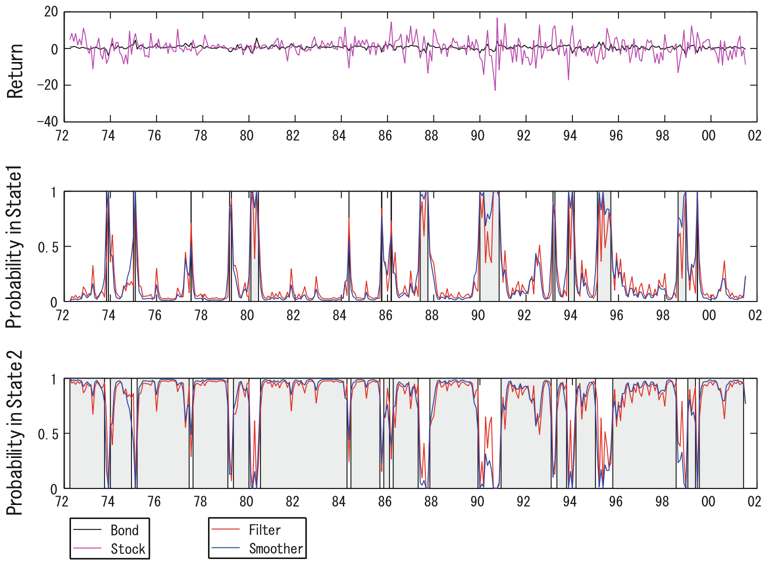

Fig. 1 Returns, filtered and smoothed probabilities in the two regime estimation. *Shaded areas* show that the corresponding periods have the smoothed probability above 0.5

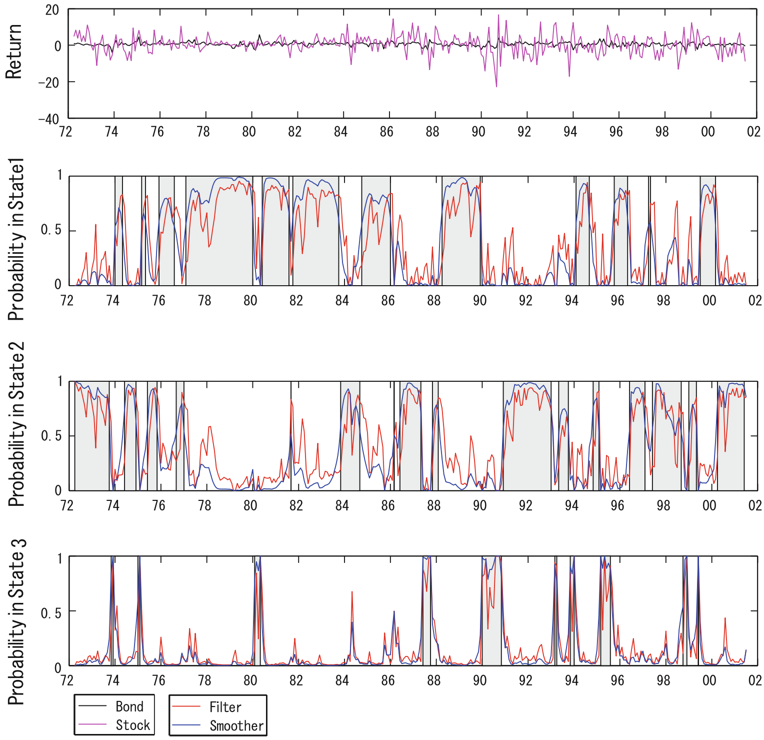

Fig. 2 Returns, filtered and smoothed probabilities in the three regime estimation. *Shaded areas* show that the corresponding periods have the smoothed probability above 0.5

Table 4 Official business cycle expansion and contraction datings from the Economic and Social Research Institute (http://www.esri.cao.go.jp/index-e.html)

Cycle	Dates			Cycle durations (months)		
	Trough	Peak	Trough	Expansion	Recession	Total cycle
1	Jun-51	Oct-51			4	
2	Oct-51	Jan-54	Nov-54	27	10	37
3	Nov-54	Jun-57	Jun-58	31	12	43
4	Jun-58	Dec-61	Oct-62	42	10	52
5	Oct-62	Oct-64	Oct-65	24	12	36
6	Oct-65	Jul-70	Dec-71	57	17	74
7	Dec-71	Nov-73	Mar-75	23	16	39
8	Mar-75	Jan-77	Oct-77	22	9	31
9	Oct-77	Feb-80	Feb-83	28	36	64
10	Feb-83	Jun-85	Nov-86	28	17	45
11	Nov-86	Feb-91	Oct-93	51	32	83
12	Oct-93	Mar-97	Apr-99	43	20	63
13	Apr-99	Oct-00		21		

The average durations for the expansions and recessions are 33.1 and 17.4 months respectively

The estimated parameters for the covariance suggest that the covariance is also regime-dependent. For a better understanding, we scale the covariance to the correlation coefficient. The correlation is 0.106 in the first state and 0.135 in the second state. The difference is almost 30%, and we consider that an introduction of time-varying covariance should play a more important role in the portfolio selection theory. This becomes clear in the three state-model, where the difference is much more significant.

One interesting result in the Fig. 1 is that many of the 'bad' market conditions are triggered by the incidents in the real economic sector. This confirms the results of Perez-Quiros and Timmermann (1998) who observed that the volatility of returns increases at the turning points of the business cycle. For example, the recessions caused by the two energy crisis's in 1973–1975 and 1979–1983, the high-Yen recession after the Plaza agreement in September 1985 and the end of the 'bubble' economy in the beginning of 1990, seem to have caused the market turbulence. Notice that this turbulent state also corresponds to an economic trough, for example, in the beginning of 1974, the end of 1986 and the end of 1993.

Next, we analyze the result from the three-state model. From the estimated parameters in the Table 1 we can interpret that the third state corresponds to the 'bad' market condition. This state and the first state in the previous model can be considered to represent the same states as the parameters and the filtered and smoothed probabilities are similar. The first and the second state in this model shows the 'normal' market conditions, the former with higher stock returns and the latter with higher bond returns. We can conjecture that the second state in the previous model corresponds to the combination of the first and second state in this model. In other words, the second state in the previous model is divided into two parts in this model: One with higher stock returns and the other with higher bond returns. Combining with the result for the expected durations, we can interpret the result that the market stays at the 'normal' state for the most of the time, though the turbulent 'bad' state arrives occasionally.

Another interesting fact is clear from the transition probability matrix. The transition probability matrix shows that while the probability of moving from the first state to the second is three times higher than that of moving to the third, the probability of moving from the second to the third is also significantly higher than the probability of moving to the first. In other words, when the bonds are performing better than stocks, it is likely that the market is heading to the turbulent state. Therefore, it may not be a good idea to invest heavily in stocks during the second state. This observation will be important in Sect. 5, where the asset allocation simulation is conducted.

Finally, the covariances are worth noting. As in the two state model, we calculate the correlation coefficient in each state: 0.09 in the first state, 0.19 in the second state and 0.12 in the third state. The difference is much more significant than the ones in the two-state model and the importance of the *regime-switching covariance* should be clear in the context of diversification.

4.3 Expanding-Window Estimation

In order to test whether our model is practical, we conduct an expanding-window estimation to investigate the stability of coefficients. In estimating MS models, we must pay attention to avoid the problem of obtaining unreliable coefficients caused by using too few data. Since our model requires a relatively large data set to be reliable, we are able to conduct the test for only last 30 months. That is, we use the first 322 data points to estimate the model parameters, and then repeat the estimation 30-times by increasing the number of sample one by one. Since we have total of 352 data points, we obtain 30 sets of estimated parameters. Figure 3 shows the results.

All coefficients are stable for the most of the time, excluding the period of the year 2000. During this period, the bond return in the second and the third state goes up and stay constant, while it stays low in the first state. The change in the stock market is subtler, showing a slight upward shift in the first and the third states. Changes also occur in volatility estimates, showing declines during this period. The most obvious change occurs in the bond-stock correlation, where some unusual upward shift is observed. Although the reason is not clear, the 'zero interest rate policy' by Bank of Japan, which has lasted since February 1999 to August 2000, may have caused this unusual return change.

5 Asset Allocation Simulation

In this section, we conduct an asset allocation simulation using the results obtained in the previous chapter. The simulation is conducted in two ways. The first one is a naive implementation, where the problem (40) is solved with a constant $\bar{\sigma}_P^2$. However, the performance of this method is disappointing, since it weighs too much on the stock when the stock market is in bad condition and consequently, the portfolio suffers from large losses during the bear market. To avoid this problem, we introduce

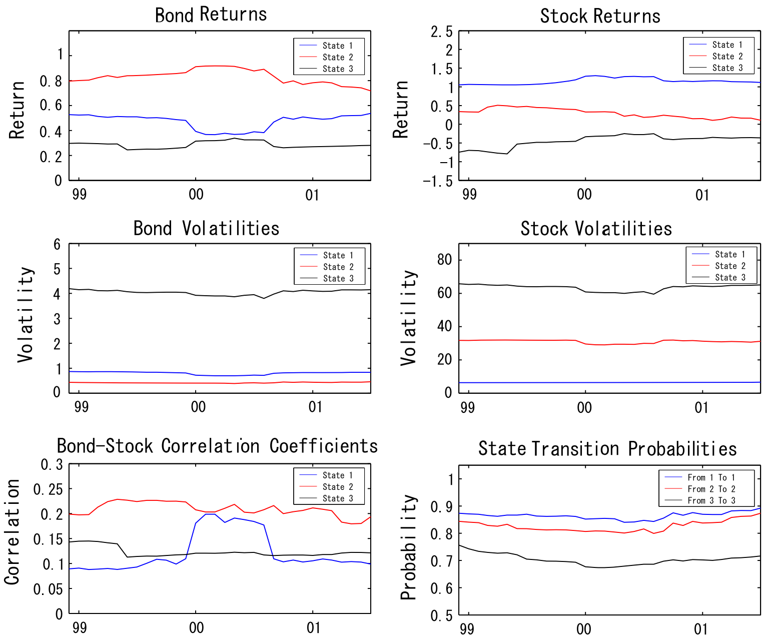

Fig. 3 Estimated parameters in the three regime, expanding window estimation

Table 5 The result for the asset allocation simulation under three regimes

	Bond-only	Stock-only	Target volatility			
			1	3	5	10
Mean	0.332	0.187	0.307	0.314	0.302	0.303
Variance	0.663	28.015	0.734	3.209	5.743	12.034

The target volatilities are constant

an idea of what we call 'switching volatility targets,' where the target volatility $\bar{\sigma}_P^2$ is set according to the states beliefs at each period. That is, the model weighs more on stocks when they are in a good state, and vice versa. The portfolio with this modified method performs much better than the one with constant volatility target.

5.1 Implementation with a Constant Volatility Target

We calculated the portfolio optimization problem of (40) using the sequential estimation results from the previous section. For the target volatility, $\bar{\sigma}_P^2$, we selected 1, 3, 5 and 10%. Table 5 shows the results. The second and third columns, 'Bond-Only' and 'Stock-Only' show the mean and variance of the bond and stock indices alone. The fourth to seventh column show the performance of the regime switching portfolios, with target volatilities of 1, 3, 5 and 10%.

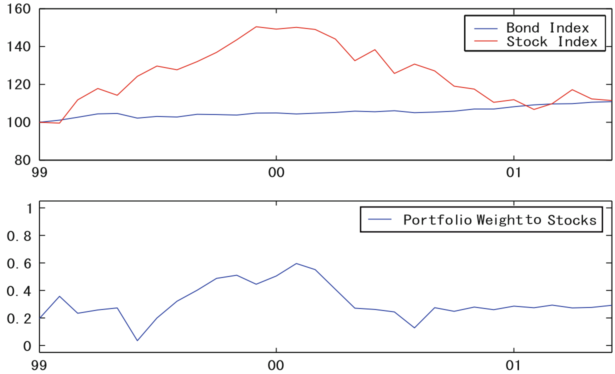

Fig. 4 The actual bond and stock index and the portfolio weight to stocks. The case with a fixed target volatility

Overall, the performances of the regime switching portfolios are disappointing. Although the means are higher than stock index, the bond index has much higher mean and lower variance. Moreover, the result shows that increasing the target volatility, meaning taking more risk, does not necessarily increase the expected return.

The reason for such low performance is clear when we look at Fig. 4. This shows the relationship between the bond and stock indices' returns and the portfolio weight to stocks. Although the portfolio invests largely in stocks during the bull market of 1999, the weight in stocks does not decline much during the bear market of 2000. Obviously this caused losses in the portfolio value during the bear market. We can conclude that, although the portfolio weight is moving to right direction depending on the market condition, it does not move enough to successfully time the market. Since the constant target volatility is the main reason for this poor performance, the next section considers varying it depending on each state probability.

5.2 Implementation with Switching Volatility Targets

The disappointing result from the previous analysis is caused by too much risk taking during the second and the third regime. Then, it would be a reasonable idea to take more volatility risk during the first state and less in the second and the third state. We call this a *switching target volatility* and define $\bar{\sigma}_P^2 = \sum_{k=1}^{K} \Pr(Y_t = e_k) \cdot \sigma_k^2$. Here σ_k^2 shows the target volatility for the state e_k. So $\bar{\sigma}_P^2$ is the state probability-weighted average of the target volatilities in each regime.

Table 6 shows the asset allocation simulation using this state-dependent target volatility. Four different target volatilities are tested. '5-1-0.5' in the table means the target volatility is 5, 1 and 0.5 for regime one, two and three respectively. In this way, '5-1-0.5', '10-1-0.5', '5-0.5-0.1' and '10-0.5-0.1' are tested. The target volatility is set this way to take into account the result from last section regarding the transition probability matrix. Since in the first state, the stocks are expected to grow with low

Table 6 The result for the asset allocation simulation under three regimes

	Bond-only	Stock-only	Target volatility			
			5-1-0.5	10-1-0.5	5-0.5-0.1	10-0.5-0.1
Mean	0.332	0.187	0.344	0.324	0.373	0.340
Variance	0.663	28.015	3.230	6.158	3.029	5.928

The target volatilities are state dependent. For example, '5-1-0.5' means the target volatilities are 5, 1 and 0.5 for regime one, two and three respectively

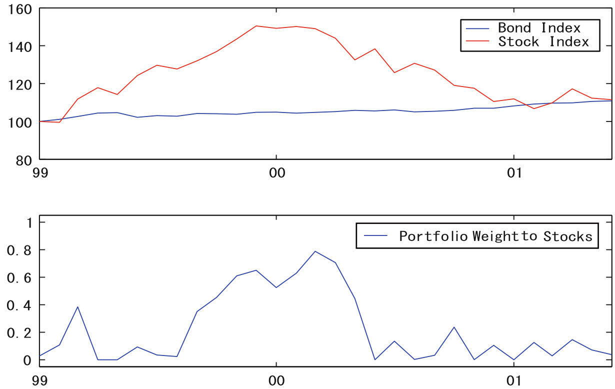

Fig. 5 The actual bond and stock index and the portfolio weight to stocks. The case with a state-dependent target volatility

volatility, the portfolio takes more risk. It does not take much risk in the second state because the probability of going into the third state cannot be ignored. In the third state, the portfolio tries to take little risk as possible. If we select a very low target volatility, the inequality constraint in 40 may not be satisfied. In this case, we select the portfolio weight that achieves the lowest volatility, which is equivalent to calculating the minimum variance portfolio.

Overall, the portfolio with the state-dependent target volatility performs better than the naive portfolio. The target volatility of '5-0.5-0.1' shows the best performance among the four. The mean of the portfolio return is improved by $0.06 \sim 0.07\%$ monthly or $0.72 \sim 0.84\%$ annually. However, our portfolio does not significantly outperform the bond index. This is caused by the highly disappointing performance by the Japanese stock market in 1990s.

Figure 5 shows the relationship between security returns and the portfolio weight to stocks. Note the obvious difference in the weight. The portfolio weights more on stocks during the bull stock markets and less during the bear market. Moreover, the weight of stocks during the bull market is much higher than the one with fixed target volatility. The opposite is true during the bear market. We can say that the state beliefs are reliable indicators to time the market.

6 Conclusion

In this paper, we have developed and implemented the theory of portfolio selection under regime switching means and volatilities that are modeled by using a Markov switching model. Although the intertemporal portfolio selection theory in general is difficult to solve, we made it significantly easier by using the log-mean variance criteria.

The estimation of the security return model was successful, yielding several interesting insights. This includes the relationship between security markets and business cycle, the expected duration of bull and bear markets, determining the probability of which security market would perform better. This kind of rich information cannot be obtained in a normal, one-state model. Also, the filtered probabilities proved to be reliable indicators to time the market.

However the asset allocation simulation was not as good as we expected. Although the performance is improved by introducing an idea of a state-dependent target volatility, the results were not surprisingly good. This is caused by the highly disappointing performance by the Japanese stock market in 1990s. We hope that during a period with occasional bull and bear markets, our model would perform much better and prove the importance of a regime switching portfolio.

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