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Log Mean-Variance Portfolio Selection Under Regime Switching

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Abstract In this paper we develop a portfolio selection theory considering discrete regime shifts in the investment opportunity and conduct an empirical analysis using Japanese sector indices to verify its effectiveness. Specifically, we model the regime shifts using a first-order Markov switching model and consider a dynamic portfolio selection problem using log mean-variance criteria. The estimation result implies that the model allows us to extract stable regimes when the number of regimes is appropriately chosen. Taking advantage of these regimes, we can improve the performance of the portfolio with rebalancing frequencies kept low.

Keywords Regime switching model · Dynamic portfolio selection · Discrete-time · Log mean-variance criteria · Quadratic programming · EM algorithm

1 Introduction

In this paper we develop a discrete-time optimal portfolio selection theory under regime switching investment opportunities. We also conduct an empirical analysis after explaining how to estimate the regimes using the actual market data.

We assume that the regime switching occurs discretely and also the asset price fluctuation is discrete. The model adopted in this paper is called the basic Markov

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switching model. Hamilton (1989) developed this model using a first-order Markov process and time-homogeneous transition probability matrix. The model describes the asset price process with a drift term that shows the expected value and a diffusion term that shows the variance and admits them to switch according to regimes.

The Markov switching model is one of the most intensively studied area in finance. For example, Timmermann (2000) confirms that the model is able to realize a wide variety of moments and serial correlations even with a simple first-order AR model. Duan et al. (2002) consider the pricing problem of European call option under regime switching with time-varying transition probability matrix. They show that a regime switching model with time-varying transition probability matrix dominates GARCH model as its special case. Therefore, although this paper adopts the basic model, it is very likely to be able to consider much more general specification.

This paper considers a problem of discrete-time portfolio selection with discrete regime switching. In the basic model, we assume that the log returns of each asset is serially independent, given the regime. The portfolio selection criteria is log mean-variance, where the return is defined as the mean of terminal portfolio log-value (i.e., the expected growth) and the risk is the variance of the terminal portfolio log-value (i.e., the variance of the growth).

There are three advantages adopting the log mean-variance criteria:

1. The log mean-variance criterion is justified from a view point of an economic theory when a long-term portfolio selection is considered. See Luenberger (1993) for details.
2. With the characteristics of log function, the terminal portfolio log-value can be expressed as the sum of log-return in each period. Also, the returns of each asset are serially independent given the regime. Therefore, the risk and return over the investment horizon can be expressed as the sum of risk and return in each period. As a result, the optimal portfolio can be obtained by solving a quadratic programming in each period.
3. Using the log mean-variance criteria with given regimes, it is possible to draw efficient frontiers, which is very familiar to the finance literature. Also, it is possible to select a portfolio optimally with predicted scenarios of future regimes.

This paper is composed in the following way: Section two models the asset price process given the regimes and develops the theory of optimal portfolio selection according to the log mean-variance criteria. Section three describes the modeling of regime switching dynamics and the method of regime estimation. Section four conducts a sector allocation simulation using the theory and shows its effectiveness.

2 Model

There are n assets traded in the market. The asset price is observed at discrete time t $(t = 0, 1, \ldots, T)$. The time interval $(t-1, t]$ is called the period t. The log return at the period t is defined as $\boldsymbol{R}_t = (R_{1t} \ldots R_{it} \ldots R_{nt})'$ using the asset prices $\boldsymbol{S}_{t-1}$ and $\boldsymbol{S}_t$:

$$R_{it} \triangleq \log\left(\frac{S_{it}}{S_{it-1}}\right) = \mu_{it} + \boldsymbol{\sigma}_{it}\boldsymbol{\varepsilon}_t \ (i = 1, \ldots, n),$$
$$\text{or,} \quad \boldsymbol{R}_t = \boldsymbol{\mu}_t + \boldsymbol{\Sigma}_t\boldsymbol{\varepsilon}_t, \tag{2.1}$$

where $\boldsymbol{\mu}_t = (\mu_{1t} \ldots \mu_{it} \ldots \mu_{nt})'$ is the drift parameter and $\boldsymbol{\Sigma}_t = \left(\sigma_{ij,t}\right)_{1\leq i,j\leq n} = \left(\boldsymbol{\sigma}_{1t}' \ldots \boldsymbol{\sigma}_{it}' \ldots \boldsymbol{\sigma}_{nt}'\right)'$ is the diffusion parameter with $\boldsymbol{\sigma}_{it}$ showing the ith row of the diffusion matrix. $\boldsymbol{\varepsilon}_t \sim N_n(\mathbf{0}, \mathbf{I})$ is the innovation term with the superscript $'$ showing the transpose.

We assume that there exists K number of regimes in the economy and the drift $\boldsymbol{\mu}_t$ and the diffusion $\boldsymbol{\Sigma}_t$ switch according to the regime at time $t(t = 1, \ldots, T)$. The regime is assumed to be described by a first-order Markov process and denoted using a K-dimensional vector Y_t with its state space $\{e_1, \ldots, e_k, \ldots, e_K\}$, where e_k is a K-dimensional vector with the kth element equal to one and zero otherwise. The filtration is $\mathcal{F}_t^Y \triangleq \sigma(Y_u;\ u = 1, \ldots, t)$. We assume that the drift parameter $\boldsymbol{\mu} = \{\boldsymbol{\mu}_t;\ t = 1, \ldots, T\}$ and the diffusion parameter $\boldsymbol{\Sigma} = \{\boldsymbol{\Sigma}_t;\ t = 1, \ldots, T\}$ has K realizations, that is, $\{\boldsymbol{\mu}(1), \ldots, \boldsymbol{\mu}(K)\}$ and $\{\boldsymbol{\Sigma}(1), \ldots, \boldsymbol{\Sigma}(K)\}$ respectively. Here $\boldsymbol{\Sigma}(k) = \left(\boldsymbol{\sigma}_1'(k) \ldots \boldsymbol{\sigma}_i'(k) \ldots \boldsymbol{\sigma}_n'(k)\right)'$ $(k = 1, \ldots, K)$. Also define an indicator function:

$$I_t(k) \triangleq \langle Y_t, e_k\rangle = \begin{cases} 1 & (\text{if } Y_t = e_k) \\ 0 & (\text{otherwise}). \end{cases} \tag{2.2}$$

The operator $\langle , \rangle$ denotes the inner product. With these notations, the drift and diffusion parameters are described as

$$\boldsymbol{\mu}_t = \sum_{k=1}^{K} I_t(k)\boldsymbol{\mu}(k) \triangleq \boldsymbol{\mu}(Y_t), \tag{2.3}$$

$$\boldsymbol{\Sigma}_t = \sum_{k=1}^{K} I_t(k)\boldsymbol{\Sigma}(k) \triangleq \boldsymbol{\Sigma}(Y_t). \tag{2.4}$$

Also, we abbreviate

$$\boldsymbol{\Lambda}(k) \triangleq \boldsymbol{\Sigma}(k)\boldsymbol{\Sigma}(k)',$$
$$\boldsymbol{\lambda}(k) \triangleq \left(\boldsymbol{\sigma}_1(k)\boldsymbol{\sigma}_1(k)' \ldots \boldsymbol{\sigma}_i(k)\boldsymbol{\sigma}_i(k)' \ldots \boldsymbol{\sigma}_n(k)\boldsymbol{\sigma}_n(k)'\right)',$$

and we simplify the notations as follows:

$$\boldsymbol{\Lambda}_t = \boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t' = \sum_{k=1}^{K} I_t(k)\boldsymbol{\Sigma}(k)\boldsymbol{\Sigma}(k)' \triangleq \boldsymbol{\Lambda}(Y_t),$$

$$\boldsymbol{\lambda}_t = \left(\boldsymbol{\sigma}_{1t}\boldsymbol{\sigma}_{1t}' \ldots \boldsymbol{\sigma}_{it}\boldsymbol{\sigma}_{it}' \ldots \boldsymbol{\sigma}_{nt}\boldsymbol{\sigma}_{nt}'\right)' = \sum_{k=1}^{K} I_t(k)\boldsymbol{\lambda}(k) \triangleq \boldsymbol{\lambda}(Y_t).$$

Thus we can express the log-return (2.1) as a process conditional on the regimes:

$$\boldsymbol{R}_t \,|\, Y_t = \boldsymbol{\mu}(Y_t) + \boldsymbol{\Sigma}(Y_t)\boldsymbol{\varepsilon}_t \quad (t = 1, \ldots, T). \tag{2.5}$$

The investor constructs the portfolio at the beginning of period t. The portfolio weight is denoted as $\boldsymbol{b}_t = (b_{1t} \ldots b_{it} \ldots b_{nt})'$. That is, b_{it} shows the proportion of ith asset in the entire portfolio value at the beginning of the period t and is held through the period t. The investor selects the portfolio within the following feasible region $\mathbf{D}$:

$$\mathbf{D} \triangleq \left\{ \boldsymbol{b} \in \mathbb{R}^n \,\middle|\, \boldsymbol{b}'\mathbf{1} = 1,\ \boldsymbol{b} \geq \mathbf{0},\ \mathbf{A}\boldsymbol{b} \leq \boldsymbol{c} \right\}, \tag{2.6}$$

where, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{c} \in \mathbb{R}^m$ and $\mathbf{1}$ is a column vector with all elements equal to one. We consider the last inequality to meet practical requirements.

With the above settings, we adopt the "*Regime Switching Log Mean-variance criteria*" for the portfolio selection:

Regime Switching Log Mean-variance is a criterion that maximizes the expected log-mean of the terminal portfolio value at the end of the investment horizon (return), considering the constraint on its log-variance (risk).

Many authors have studied the characteristics of log mean-variance criterion. Luenberger (1993) proves that the log mean-variance is an adequate criterion from economic theory point of view when considering a long term investment horizon. Konno et al. (1993) develop an efficient algorithm to calculate the efficient frontiers. Also, Ishijima and Uchida (2010) study the Merton's problem under regime switching. Using the log-linear approximation (Campbell and Viceira 2002), the portfolio log return with the given regime at period t is

$$R_t^P | Y_t \triangleq \log \left(\frac{V_t}{V_{t-1}} \right) \Bigg| \, Y_t = \boldsymbol{b}_t' \boldsymbol{R}_t | Y_t + \frac{1}{2} \boldsymbol{b}_t' \boldsymbol{\lambda}(Y_t) - \frac{1}{2} \boldsymbol{b}_t' \boldsymbol{\Lambda}(Y_t) \boldsymbol{b}_t \tag{2.7}$$

$$= \mu_P(\boldsymbol{b}_t;\ Y_t) + \boldsymbol{b}_t' \boldsymbol{\Sigma}(Y_t) \boldsymbol{\varepsilon}_t, \tag{2.8}$$

where

$$\mu_P(\boldsymbol{b}_t;\ Y_t) \triangleq \boldsymbol{b}_t' \boldsymbol{\mu}(Y_t) - \frac{1}{2} \boldsymbol{b}_t' \boldsymbol{\Lambda}(Y_t) \boldsymbol{b}_t + \frac{1}{2} \boldsymbol{b}_t' \boldsymbol{\lambda}(Y_t). \tag{2.9}$$

The log-linear approximation is applied in (2.7). By considering the approximation, the log return of the portfolio value can be expressed as an weighted sum of the log return of each asset. Also, we call $\mu_P(\boldsymbol{b}_t;\ Y_t)$ the "regime conditional portfolio return" and this is the return measure in this paper.

Assuming $V_0 = 1$, the log portfolio value given the regime can be expressed as:

$$\log V_T|\mathcal{F}_T^Y = \log\left(\frac{V_1}{V_0}\cdot\cdots\cdot\frac{V_T}{V_{T-1}}\right)\bigg|\mathcal{F}_T^Y = \sum_{t=1}^{T} R_t^P|Y_t \tag{2.10}$$

$$= \sum_{t=1}^{T}\left\{\mu_P(\boldsymbol{b}_t;\ Y_t) + \boldsymbol{b}_t'\boldsymbol{\Sigma}(Y_t)\boldsymbol{\varepsilon}_t\right\}. \tag{2.11}$$

Using the above notations, the expected value and the variance of the terminal portfolio value given the regime is

$$E\left[\log V_T|\mathcal{F}_T^Y\right] = \sum_{t=1}^{T}\mu_P(\boldsymbol{b}_t; Y_t), \tag{2.12}$$

$$V\left[\log V_T|\mathcal{F}_T^Y\right] = \sum_{t=1}^{T} V\left[\mu_P(\boldsymbol{b}_t;\ Y_t) + \boldsymbol{b}_t'\boldsymbol{\Sigma}(Y_t)\boldsymbol{\varepsilon}_t|\mathcal{F}_T^Y\right] \tag{2.13}$$

$$= \sum_{t=1}^{T}\sigma_P^2(\boldsymbol{b}_t;\ Y_t), \tag{2.14}$$

where

$$\sigma_P^2(\boldsymbol{b}_t;\ Y_t) \triangleq \boldsymbol{b}_t'\boldsymbol{\Lambda}(Y_t)\boldsymbol{b}_t. \tag{2.15}$$

We obtain (2.13) with the assumption that $\boldsymbol{\varepsilon}_t$ are serially independent given the regime and (2.14) using $\boldsymbol{\Lambda}(Y_t) = \boldsymbol{\Sigma}(Y_t)\boldsymbol{\Sigma}(Y_t)'$. We call $\sigma_P^2(\boldsymbol{b}_t;\ Y_t)$ the "regime conditional portfolio risk" and this is the risk measure in our model. Using the following expression

$$\boldsymbol{\mu}_P(\boldsymbol{b}_t) = (\mu_P(\boldsymbol{b}_t;\ e_1)\ldots\mu_P(\boldsymbol{b}_t;\ e_k)\ldots\mu_P(\boldsymbol{b}_t;\ e_K)), \tag{2.16}$$

$$\boldsymbol{\sigma}_P^2(\boldsymbol{b}_t) = \left(\sigma_P^2(\boldsymbol{b}_t;\ e_1)\ldots\sigma_P^2(\boldsymbol{b}_t;\ e_k)\ldots\sigma_P^2(\boldsymbol{b}_t;\ e_K)\right), \tag{2.17}$$

the regime conditional portfolio return $\mu_P(\boldsymbol{b}_t;\ Y_t)$ and risk $\sigma_P^2(\boldsymbol{b}_t;\ Y_t)$ can be written as

$$\mu_P(\boldsymbol{b}_t;\ Y_t) = \boldsymbol{\mu}_P(\boldsymbol{b}_t)Y_t, \tag{2.18}$$

$$\sigma_P^2(\boldsymbol{b}_t;\ Y_t) = \boldsymbol{\sigma}_P^2(\boldsymbol{b}_t)Y_t. \tag{2.19}$$

We obtain the unconditional expected value and variance of the log portfolio value by taking an expectation with respect to the regime Y:

$$E_0\left[\log V_T\right] = E_0\left[E\left[\log V_T|\mathcal{F}_T^Y\right]\right] = \sum_{t=1}^{T}\boldsymbol{\mu}_P(\boldsymbol{b}_t)E_0\left[E\left[Y_t|Y_{t-1}\right]\right] \tag{2.20}$$

$$= \sum_{t=1}^{T}\boldsymbol{\mu}_P(\boldsymbol{b}_t)\bar{Y}_t, \tag{2.21}$$

$$V_0\left[\log V_T\right] = E_0\left[\ V\left[\log V_T|\mathcal{F}_T^Y\right]\right] = \sum_{t=1}^{T}\boldsymbol{\sigma}_P^2(\boldsymbol{b}_t)E_0\left[E\left[Y_t|Y_{t-1}\ \right]\right] \quad (2.22)$$

$$= \sum_{t=1}^{T}\boldsymbol{\sigma}_P^2(\boldsymbol{b}_t)\bar{Y}_t. \quad (2.23)$$

Here,

$$\bar{Y}_t \ \triangleq \sum_{k=1}^{K}\xi_{k,t|t-1}e_k, \quad (2.24)$$

$$\xi_{k,t|t-1} \ \triangleq \Pr\left(Y_t = e_k|Y_{t-1}\right), \quad (2.25)$$

and $\xi_{k,t|t-1}$ is the "regime inference" at time t conditional on the information at time $t-1$. Using this expression, we can write the unconditional expected value and the variance of the log terminal value as

$$E_0\left[\ \log V_T\ \right] = \sum_{t=1}^{T}\sum_{k=1}^{K}\xi_{k,t|t-1}\mu_P(\boldsymbol{b}_t;\ e_k), \quad (2.26)$$

$$V_0\left[\log V_T\right] = \sum_{t=1}^{T}\sum_{k=1}^{K}\xi_{k,t|t-1}\sigma_P^2(\boldsymbol{b}_t;\ e_k). \quad (2.27)$$

Equation (2.26) means that the expected value of the terminal portfolio log-value is obtained by summing the regime conditional portfolio returns $\mu_P(\boldsymbol{b}_t;\ e_k)$ weighted by the regime inference $\xi_{k,t|t-1}$ and then again summing up with respect to t. Similarly, (2.27) means that the variance of the terminal portfolio log-value is obtained by summing the regime condtinal portfolio risk $\sigma_P^2(\boldsymbol{b}_t;\ e_k)$ weighted by the regime inference $\xi_{k,t|t-1}$ and then again summing up with respect to t.

Now we are ready to formulate the portfolio selection problem based on the regime switching log mean-variance criteria. We call the tolerable portfolio risk a "target volatility" and denote by $\bar{\sigma}^2$. The optimal portfolio $\boldsymbol{b} = \{\boldsymbol{b}_t;\ t = 1, \ldots, T\}$ in the feasible region $\mathbf{D}$ in (2.6) is the solution of the following problem $\mathbf{P}$:

$$\mathbf{P}\left|\begin{array}{ll} \underset{\boldsymbol{b}}{\text{maximize}} & E_0\left[\ \log V_T\ \right] = \sum_{t=1}^{T}\sum_{k=1}^{K}\xi_{k,t|t-1}\mu_P(\boldsymbol{b}_t;\ e_k) \\ \text{subject to} & V_0\left[\ \log V_T\ \right] = \sum_{t=1}^{T}\sum_{k=1}^{K}\xi_{k,t|t-1}\sigma_P^2(\boldsymbol{b}_t;\ e_k) = \bar{\sigma}^2\cdot T, \\ & \boldsymbol{b}_t \in \mathbf{D}\ (t = 1, \ldots, T). \end{array}\right. \quad (2.28)$$

In fact, solving the problem $\mathbf{P}$ is equivalent to solving the following quadratic programming problem $\mathbf{P}_t$ $(t = 1, \ldots, T)$ at the beginning of the each period:

$$\mathbf{P}_t \left| \begin{array}{ll} \underset{\boldsymbol{b}_t}{\text{maximize}} & \sum_{k=1}^{K} \xi_{k,t|t-1} \mu_P(\boldsymbol{b}_t;\ e_k) \\ \text{subject to} & \sum_{k=1}^{K} \xi_{k,t|t-1} \sigma_P^2(\boldsymbol{b}_t;\ e_k) = \bar{\sigma}^2, \\ & \boldsymbol{b}_t \in \mathbf{D}. \end{array} \right. \tag{2.29}$$

We can obtain an efficient frontier on a plane with portfolio risk on the x-axis and portfolio return on the y-axis, by solving the problem $\mathbf{P}_t$ with various target volatilities. See Konno et al. (1993) for details. This kind of efficient frontier generalizes the one in a Markowitz's mean-variance sense, considering the intertemporal investment and the existence of regimes in the investment opportunity.

The two extreme points on the efficient frontier for regime switching log mean-variance criteria are the log-mean maximizing portfolio (portfolio K) and the log-variance minimizing portfolio (portfolio M). They are the solutions of the following problems $\mathbf{K}_t$ $(t = 1, \ldots, T)$ and $\mathbf{M}_t$ $(t = 1, \ldots, T)$:

$$\mathbf{K}_t \left| \begin{array}{ll} \underset{\boldsymbol{b}_t}{\text{maximize}} & \sum_{k=1}^{K} \xi_{k,t|t-1} \mu_P(\boldsymbol{b}_t;\ e_k) \\ \text{subject to} & \boldsymbol{b}_t \in \mathbf{D}, \end{array} \right. \tag{2.30}$$

$$\mathbf{M}_t \left| \begin{array}{ll} \underset{\boldsymbol{b}_t}{\text{minimize}} & \sum_{k=1}^{K} \xi_{k,t|t-1} \sigma_P^2(\boldsymbol{b}_t;\ e_k) \\ \text{subject to} & \boldsymbol{b}_t \in \mathbf{D}. \end{array} \right. \tag{2.31}$$

3 Regime Dynamics and its Estimations

3.1 Regime Dynamics

We model the dynamics of the reimge Y using the first-order Markov switching model in the previous section. The time-homogeneous transition probability of shifting from the regime e_l at time t to the regime e_k at time $t+1$ is denoted by

$$\mathbf{P} \triangleq (p_{kl})_{1 \le k,l \le K} = \Big(\Pr\left(Y_{t+1} = e_k \,|Y_t = e_l\,\right) \Big)_{1 \le k,l \le K}. \tag{3.32}$$

Writing $\mathcal{F}_t^Y \triangleq \sigma(Y_1, \ldots, Y_t)$, the conditional expectation of the regime is

$$\begin{aligned} E\left[\, Y_{t+1}|\mathcal{F}_t^Y \,\right] &= E\left[\, Y_{t+1}|Y_t, \mathcal{F}_{t-1}^Y \,\right] = E\left[\, Y_{t+1}\,|Y_t\,\right] \\ &= \mathbf{P}Y_t, \end{aligned}$$

where we defined $M_{t+1} \triangleq Y_{t+1} - \mathbf{P}Y_t$. Then

$$E\left[M_{t+1} \middle| \mathcal{F}_t^Y \right] = E\left[Y_{t+1} - \mathbf{P}Y_t \,|Y_t \right] = \mathbf{0}.$$

Therefore, Y_t is $\mathcal{F}_t^Y$-martingale. Using this expression, the regime is modelled as a semimartingale (see Elliott et al. 1995 for details).

$$Y_{t+1} = \mathbf{P}Y_t + M_{t+1}. \tag{3.33}$$

3.2 Estimations

Let us first define the quasi-log-likelihood function of the Markov switching model. Write $\mathscr{R}_t = \{\boldsymbol{R}_1, \ldots, \boldsymbol{R}_t\}$, $\mathscr{Y}_t = \{Y_1, \ldots, Y_t\}$ and $\mathscr{F}_t = \{\mathscr{R}_t, \mathscr{Y}_t\}$. We also define the "regime conditional density function" as

$$\eta_{k,t} \triangleq f(\boldsymbol{R}_t | Y_t = e_k, \mathscr{R}_{t-1}; \boldsymbol{\Theta}) \quad (k = 1, \ldots, K), \tag{3.34}$$

and the "regime inference" as

$$\xi_{k,t|t-1} \triangleq \Pr(Y_t = e_k | \mathscr{R}_{t-1}; \boldsymbol{\Theta}) \quad (k = 1, \ldots, K). \tag{3.35}$$

Using the above expression, we can write the quasi-likelihood function as:

$$\begin{aligned} f(\mathscr{R}_T; \boldsymbol{\Theta}) &= \prod_{t=1}^{T} \sum_{k=1}^{K} f(\boldsymbol{R}_t | Y_t = e_k, \mathscr{R}_{t-1}; \boldsymbol{\Theta}) \cdot \Pr(Y_t = e_k | \mathscr{R}_{t-1}; \boldsymbol{\Theta}) \\ &= \prod_{t=1}^{T} \boldsymbol{\eta}_t' \boldsymbol{\xi}_{t|t-1} = \prod_{t=1}^{T} \mathbf{1}' \left(\boldsymbol{\eta}_t \bigodot \boldsymbol{\xi}_{t|t-1} \right), \end{aligned} \tag{3.36}$$

where the operator $\bigodot$ represents the element-by-element multiplication. Let us now move onto the estimation procedures of the parameters in the Markov switching model. This is a type of EM algorithm introduced by Dempster et al. (1977).

Estimation-Step (Regime Updating)

We consider the relationship between $\xi_{k,t|t} = \Pr(Y_t = e_k | \mathscr{R}_t; \boldsymbol{\Theta})$ and $\xi_{k,t+1|t} = \Pr(Y_{t+1} = e_k | \mathscr{R}_t; \boldsymbol{\Theta})$ $(k = 1, \ldots, K)$. Hamilton (1994) shows that the regime updating scheme is the following:

$$\boldsymbol{\xi}_{t|t} = \frac{\boldsymbol{\eta}_t \bigodot \boldsymbol{\xi}_{t|t-1}}{\mathbf{1}' \left(\boldsymbol{\eta}_t \bigodot \boldsymbol{\xi}_{t|t-1} \right)}, \tag{3.37}$$

$$\boldsymbol{\xi}_{t+1|t} = \mathbf{P} \boldsymbol{\xi}_{t|t}. \tag{3.38}$$

We specifically call $\boldsymbol{\xi}_{t|t}$ a "filtered probability." While (3.37) calculates the regime inference at time t based on the information up to time t, it is possible to calculate it using the information up to time T. This is called "smoothed probability" and can be easily obtained by using the following scheme proposed by Kim (1993):

$$\boldsymbol{\xi}_{t|T} = \boldsymbol{\xi}_{t|t} \odot \left\{ \mathbf{P}' \cdot \left[\boldsymbol{\xi}_{t+1|T} (\div) \boldsymbol{\xi}_{t+1|t} \right] \right\}, \tag{3.39}$$

where the operator $(\div)$ does the element-by-element division of vectors. This smoothed probability is required for the next Maximization-Step.

Maximization-Step

In this step, we consider maximizing the usual likelihood function $f(\mathscr{R}_T, \mathscr{Y}_T;\ \boldsymbol{\Theta})$, instead of the quasi-likelihood function $f(\mathscr{R}_T;\ \boldsymbol{\Theta})$. According to Hamilton (1990), maximizing this likelihood function is equivalent to maximizing the quasi-likelihood function and former procedure is simpler than the latter. The likelihood function is

$$f(\mathscr{R}_T, \mathscr{Y}_T; \boldsymbol{\Theta}) = \sum_{k=1}^{K} I_1(k) \Pr(Y_1 = e_k | \mathscr{F}_0; \boldsymbol{\Theta}) f(\boldsymbol{R}_1 | Y_1 = e_k; \boldsymbol{\Theta}) \times \prod_{t=2}^{T} \sum_{k=1}^{K} \sum_{l=1}^{K} I_{t-1}(l) I_t(k) \Pr(Y_t = e_k \left| Y_{t-1} = e_l; \boldsymbol{\Theta}\right) f(\boldsymbol{R}_t | Y_t = e_k; \boldsymbol{\Theta}).$$

Here the indicator function $I_t(k)$ $(k = 1, \ldots, K)$ is defined in (2.2). Taking logs on the both sides,

$$\begin{aligned}\log f(\mathscr{R}_T, \mathscr{Y}_T; \boldsymbol{\Theta}) = & \sum_{t=1}^{T} \sum_{k=1}^{K} I_t(k) \log f(\boldsymbol{R}_t | Y_t = e_k; \boldsymbol{\Theta}) \\ & + \sum_{k=1}^{K} I_1(k) \log \Pr(Y_1 = e_k | \mathscr{F}_0; \boldsymbol{\Theta}) \\ & + \sum_{t=2}^{T} \sum_{k=1}^{K} \sum_{l=1}^{K} I_{t-1}(l) I_t(k) \log \Pr(Y_t = e_k | Y_{t-1} = e_l; \boldsymbol{\Theta}).\end{aligned}$$

Here we take an expectation under a measure defined by the paramters $\boldsymbol{\Theta}^{(j)}$, not the measure defined by $\boldsymbol{\Theta}^{(j+1)}$ that specify the log-likelihood function $\log f(\mathscr{R}_T, \mathscr{Y}_T;\ \boldsymbol{\Theta}^{(j+1)})$:

$$Q(\boldsymbol{\Theta}^{(j+1)}; \boldsymbol{\Theta}^{(j)}, \mathscr{R}_T) \triangleq E^{\boldsymbol{\Theta}^{(j)}} \left[\log f(\mathscr{R}_T, \mathscr{Y}_T;\ \boldsymbol{\Theta}^{(j+1)}) \middle| \mathscr{R}_T \right].$$

Then

$$\begin{aligned} & Q(\boldsymbol{\Theta}^{(j+1)}; \boldsymbol{\Theta}^{(j)}, \mathscr{R}_T) \\ & = -\frac{1}{2} \sum_{t=1}^{T} \sum_{k=1}^{K} \xi_{k,t|T}^{(j)} \left[n \log 2\pi + \log \left| \boldsymbol{\Sigma}(k)^{(j+1)} \right| \right. \\ & \quad \left. + (\boldsymbol{R}_t - \boldsymbol{\mu}(k)^{(j+1)})' \left(\boldsymbol{\Sigma}(k)^{(j+1)} \right)^{-1} (\boldsymbol{R}_t - \boldsymbol{\mu}(k)^{(j+1)}) \right] \\ & \quad + \sum_{k=1}^{K} \xi_{k,1|T}^{(j)} \log \rho_k^{(j+1)} + \sum_{t=2}^{T} \sum_{k=1}^{K} \sum_{l=1}^{K} \Pr\left(Y_{t-1} = e_l, Y_t = e_k | \mathscr{R}_T; \boldsymbol{\Theta}^{(j)} \right) \log p_{kl}^{(j+1)}. \end{aligned}$$

Here $\rho_k^{(j+1)} \triangleq \Pr\left(Y_1 = e_k | \mathscr{F}_0; \boldsymbol{\Theta}^{(j+1)}\right)$ and $p_{kl}^{(j+1)} \triangleq \Pr\left(Y_t = e_k | Y_{t-1} = e_l; \boldsymbol{\Theta}^{(j+1)}\right)$. We formulate the maximization of this expected log-likelihood function

$Q(\mathbf{\Theta}^{(j+1)}; \mathbf{\Theta}^{(j)}, \mathscr{R}_T)$ subject to the constraints on the transition probabilities in the following way:

$$\left| \begin{array}{l} \underset{\mathbf{\Theta}^{(j+1)}}{\text{maximize}} \ Q(\mathbf{\Theta}^{(j+1)}; \mathbf{\Theta}^{(j)}, \mathscr{R}_T) \\ \text{subject to} \ \sum_{k=1}^{K} p_{kl}^{(j+1)} = 1 \quad (l = 1, \dots, K), \\ \qquad\qquad \sum_{k=1}^{K} \rho_k^{(j+1)} = 1. \end{array} \right. \tag{3.40}$$

We obtain the Karush–Kuhn–Tucker conditions with respect to the transition probablity:

$$p_{kl}^{(j+1)} = \frac{1}{\sum_{t=2}^{T} \Pr(Y_{t-1} = e_l | \mathscr{R}_T; \mathbf{\Theta}^{(j)})} \sum_{t=2}^{T} \Pr(Y_{t-1} = e_l, Y_t = e_k | \mathscr{R}_T; \mathbf{\Theta}^{(j)}), \tag{3.41}$$

with respect to the initial regime inference $\rho_k^{(j+1)}$

$$\rho_k^{(j+1)} = \frac{1}{\sum_{l=1}^{K} \xi_{l,1|T}^{(j)}} \xi_{k,1|T}^{(j)}, \tag{3.42}$$

with respect to the drift parameters:

$$\boldsymbol{\mu}(k)^{(j+1)} = \frac{1}{\sum_{t=1}^{T} \xi_{k,t|T}^{(j)}} \sum_{t=1}^{T} \xi_{k,t|T}^{(j)} \boldsymbol{R}_t, \tag{3.43}$$

and finally with respect to the diffusion parameters:

$$\mathbf{\Sigma}(k)^{(j+1)} = \frac{1}{\sum_{t=1}^{T} \xi_{k,t|T}^{(j)}} \sum_{t=1}^{T} \xi_{k,t|T}^{(j)} (\boldsymbol{R}_t - \boldsymbol{\mu}(k)^{(j+1)})(\boldsymbol{R}_t - \boldsymbol{\mu}(k)^{(j+1)})'. \tag{3.44}$$

4 Empirical Analysis

In this section, we estimate the Markov switching model and conduct an asset allocation simulation using the regime switching log mean-variance criteria. The data adopted here are the sector indices for precision machines and electric power/gas in Japanese stock market. We test the effectiveness of investing in the latter sector that are called ‘defensive stocks’. We first estimate using the full sample and interpret the estimated regimes. We also check the stability of the parameters. Next, using the estimated parameters we conduct an asset allocation simulation and explore the possibility of applying this model in practice.

Table 1 Summary statistics of the monthly sector returns for precision machines and electric power/gas sectors

	Mean (%)	Variance	Skewness	Kurtosis
Prec. mach.	0.435	32.876	−0.654	1.508
Power/gas	0.463	26.773	0.653	3.534

4.1 Parameter Estimation: Full Sample

Before actually estimating the parameters, let us first consider the possible occasions when the model explained in the previous section effectively works. It is obvious from our modeling that the type of switching we assume happens at the same time, in all assets. This means that we can apply this theory when one asset class performs well, the other performs badly and vice versa. We can come up with several pairs of different asset classes that have such characteristics: stocks and bonds, cyclical and defensive stocks, large-cap and small-cap stocks, value and growth stocks and so on. Since Ishijima and Uchida (2010) already consider the stock and bond asset allocation problem under regime switching, we use the cyclical and defensive stock data to see the effectiveness of the defensing.

We use the sector indices of Tokyo Stock Exchange that are provided by Nikkei NEEDS. The total sample period is 385 months, which starts in February 1970 and ends in February 2002, and the selected sectors are the electric power and the precision machines sector, which has the lowest correlation with the former sector among all sectors. Table 1 shows the summary statistics of monthly returns. The returns are about the same level, while the variance is relatively higher in the precision machines sector.

Tables 2, 3, 4 show the estimation result of the Markov switching model using the same data. Let us first analyze the two state model. In the first state, the precision machines sector has higher risk and return than the electric power sector. On the other hand in the second state, both sectors have about the same level of variance, the former has a negative return while the latter has a positive one. Figure 1 shows the returns, economic conditions, filtered and smoothed probabilities to visualize when each regimes are activated. The upper graph plots the returns of the two indices and the shadowed area shows that it is the recession period.[1] Lower two graphs show the filtered and smoothed probabilities in each regime and the shadowed periods show that they have the smoothed probability greater than 50%. We can roughly say that the precision machines sector performs well during the booms while the electric power and gas sector performs well during the recessions. We can observe from the expected duration of the second regime in the Table 2 that it is not stable.

Let us next consider the estimation results of the three state model. Table 2 shows that the precision machines sector has the lower risk and return and the electric power and gas sector has the higher return and risk in the first state. The opposite is true in the third state. On the other hand, the second state is somewhat special showing that the former has higher risk and lower return than the latter. We can also observe that the parameters of the first state in the two state model is almost identical to the ones of the

[1] We follow the official datings of business cycles published by Economic and Social Research Institute. See http://www.esri.cao.go.jp for actual datings.

Table 2 Estimated parameters for the two- and three-state model

	Two state model	Three state model
State 1		
Prec. mach. return (%)	1.994 (0.354)	0.027 (0.739)
Power/gas return (%)	−0.084 (0.256)	1.832 (0.720)
Prec. mach. variance	18.559 (2.210)	30.594 (5.063)
Power/gas variance	7.868 (0.917)	63.073 (8.301)
Covariance	2.823 (1.125)	1.899 (4.612)
Expected duration (months)	5.718	6.975
State 2		
Prec. mach. return (%)	−1.932 (0.635)	−3.663 (1.079)
Power/gas return (%)	1.293 (0.647)	1.102 (0.389)
Prec. mach. variance	45.096 (5.610)	54.640 (11.753)
Power/gas variance	54.141 (5.814)	5.694 (1.384)
Covariance	4.087 (4.139)	1.684 (2.592)
Expected duration (months)	3.790	9.941
State 3		
Prec. mach. return (%)		1.907 (0.475)
Power/gas return (%)		−0.548 (0.239)
Prec. mach. variance		20.249 (3.715)
Power/gas variance		8.969 (0.631)
Covariance		4.820 (1.319)
Expected duration (months)		8.939
Log-likelihood	−2333.59	−2308.60

Standard errors are reported in the parentheses. The expected durations for each state is calculated as $1/(1-p_{ii})$, where p_{ii} shows the probability of staying at the state i. All parameters are monthly

Table 3 Transition probability matrix for the two-state model

	State 1	State 2
State 1	0.825 (0.053)	0.175 (0.043)
State 2	0.264 (0.065)	0.736 (0.043)

Standard errors are reported in the parentheses

Table 4 Transition probability matrix for the three-state model

	State 1	State 2	State 3
State 1	0.857 (0.064)	0.000 (–)	0.143 (0.066)
State 2	0.000 (–)	0.899 (0.127)	0.101 (0.142)
State 3	0.086 (0.066)	0.025 (0.064)	0.888 (0.083)

Standard errors are reported in the parentheses

third state in the three state model. We can say that the second state in the two state model is divided into two states in the three state model and became the first and the second state.

As we can see in the expected durations estimates, the regimes in the three state model are stable, which is also obvious in Fig. 2. It is clear that the period when the precision machines sector performs well (third state) corresponds to the economic

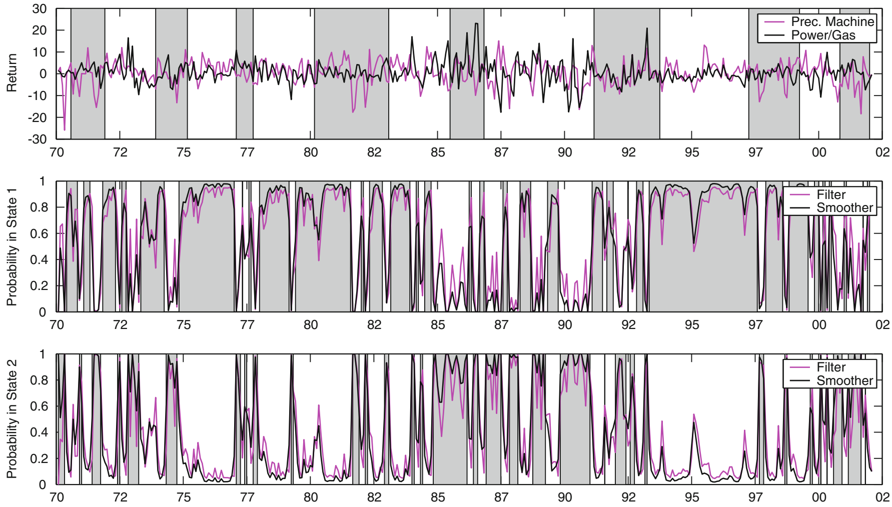

Fig. 1 Returns, filtered and smoothed probabilities in the two-state model. The *shaded areas* in the first-row figure show that the corresponding periods are in the recessions. The *shaded areas* in the second- and third-row figures show that the corresponding periods have the smoothed probability above 0.5 for each state

expansion. This implies that appropriate choice of the number of regimes is important for obtaining stable estimates. We can also observe that the electric power and gas sector, the defensive stocks, perform well during the recessions and especially the importance of the second state should be noted. This state rarely occurs, but corresponds to the big events in the economy. For example, the Asian financial crisis in 1997, the collapse of the 'IT bubble' in 2000 and 2001 and so on. This result means that the defensive stocks display their real ability during this kind of turbulent economy yielding stable returns for 10 months in average.

4.2 Parameter Estimation: Expanding Window

While the analysis in the previous section is based on the full sample available, the asset management simulation in the next section requires the out-of-sample estimation. We adopt the expanding window estimation procedure, which starts the estimation by using only a part of the whole data set and then sequentially estimating by increasing the number of sample one by one. We use this method since nonlinear models like Markov switching model require larger data samples than the linear ones (Psaradakis and Sola 1998).

We conduct the sector allocation simulation for 60 months, so we start the estimation using the data from the 1st to the 325th. The next estimation is based on from the 1st to the 326th, and we continue this until we reach the end of the data set. At the end of the all sequential estimation, we obtain 60 sets of parameters that is based on the only information available at the corresponding period. Figure 3 shows the time-series of the estimated parameters in the three state model. The result shows that

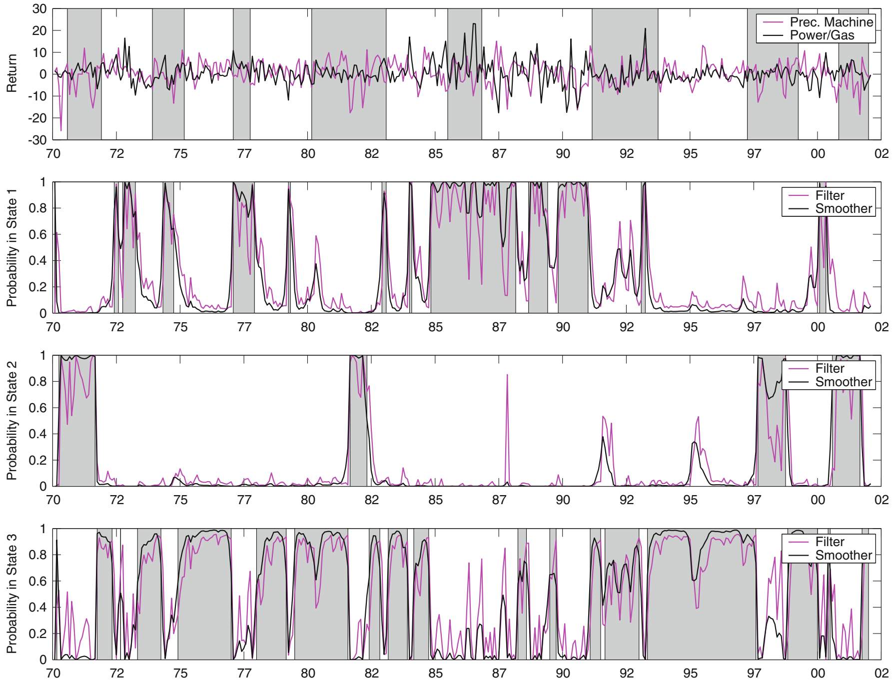

Fig. 2 Returns, filtered and smoothed probabilities in the three-state model. The *shaded areas* in the first-row figure show that the corresponding periods are in the recessions. The *shaded areas* in the second-, third- and fourth-row figures show that the corresponding periods have the smoothed probability above 0.5 for each state

almost all parameters are stably estimated. Especially the stability of the estimated returns for such a long period is notable, since it is well known that the estimation of stable returns is difficult. Also, the persistence of the high transition probabilities are worth mentioning. As will be seen in the later section, this persistence at high level results in stable estimation of regimes, and this turns out to be the trick to keep the rebalancing frequencies low.

4.3 Sector Allocation Simulation

In this section, we conduct the sector allocation simulation. We pay extra attention not to include the hindsight information. The simulation procedure is the following:

1. Estimate the parameter using the information available up to time t. For the performance comparison, we estimate two sets of parameters, with and without regimes.
2. Optimize the portfolio based on the estimated parameters. We hold this portfolio until time $t+1$. The criteria used for the optimization are: log mean maximization, log variance minimization and regime switching log mean-variance with target volatilities.
3. Repeat the procedure by expanding the sample period.

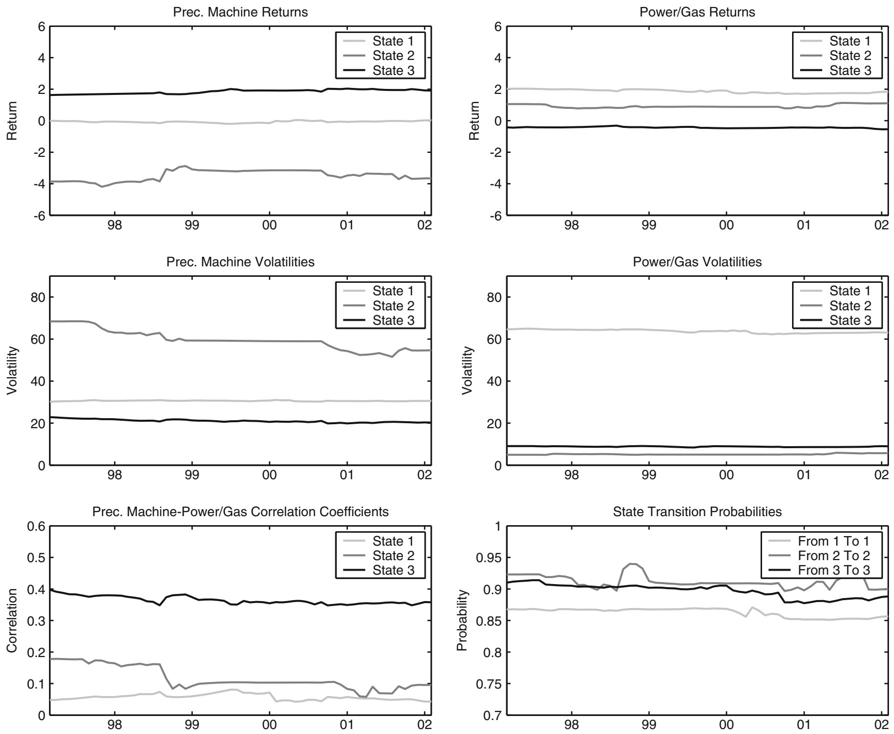

Fig. 3 Estimated parameters in the three-state model; expanding window estimation

Since the simulation is conducted for 60 months, the simulation period starts in March 1997 and ends in February 2002. Left side of Table 5 shows the mean and variance of the two indices during this period. For later comparisons, let us first analyze the results with no regimes in the right side of Table 5. The sample periods for estimating the mean and variance are preceding 3 months, 1 and 5 years. The result shows that the 3-month portfolio has an outstanding performance. This is because we are actually implementing a momentum strategy by considering such a short period of data. Although the return is high, the rebalancing frequency is also high. Let us calculate the variance of the portfolio weight as a convenient measure for the rebalancing frequency. For the log mean maximizing portfolios, the value is 12.839 for 3 months, 5.361 for 1 year and 0.597 for 5 years. For the log variance minimizing portfolios, the value is 0.088, 0.023 and 0.006 respectively. We can observe that the 3-month portfolio with log mean maximization has higher rebalancing frequencies than others. We treat this portfolio as unrealistic and omit from the further analysis. The next best performing portfolio is the one with log variance minimization and again with the sample period of 3 months. This is much more realistic with little rebalancing. It is worth noting that it is possible to achieve positive return by combining negative return assets. Other portfolios with long sample periods have disappointing results. This is because they are too much dragged by the information from long time ago and cannot follow the current market conditions to generate positive returns.

Table 5 Mean and variance of the simulated portfolio's monthly return without regime switching

	Sector returns		Log-mean max. portfolio			Log-variance min. portfolio		
	Prec. mach.	Power/gas	3 Months	1 Year	5 Years	3 Months	1 Year	5 Years
Mean (%)	−0.272	−0.057	0.509	−0.61	−0.315	0.040	−0.270	−0.308
Variance	36.786	12.738	12.544	10.304	8.423	11.349	8.914	8.539

The results for both the log-mean maximizing and the log-variance minimizing portfolios are reported by varying sample periods

Table 6 Mean and variance of the simulated portfolio's monthly return with regime switching

	Log-mean max. portfolio	Log-variance min. portfolio	Target volatility		
			20% Portfolio	30% Portfolio	40% Portfolio
Mean (%)	0.024	0.068	−0.254	−0.608	−0.712
Variance	10.099	10.044	18.745	22.496	24.980

Table 6 shows the result with regimes. The log mean maximizing and the log variance minimizing portfolios yield almost identical result. They provide higher returns and lower volatilities than the ones without considering regimes. Moreover, the rebalancing frequency measure for this portfolio is 0.021, which is about one-fourth. This means that this portfolio generate higher returns and lower risks with low rebalancing frequencies. We can consider this as the best performing portfolio in our analysis.

Table 6 shows the results with target volatilities but they are disappointing. This is the result of taking too much volatility risk where the risk should have been kept low. The portfolio with little risk, the log variance minimizing portfolio was the wise choice for this period.

We can conclude that we can improve the performance of the portofolio with low rebalancing frequencies by considering the regimes. To take advantage of the defensive stocks, it is important to optimize the portfolio considering the switching in the investment opportunities.

5 Conclusion

We propose a model of dynamic portfolio selection subject to regime shifts in the investment opportunity set under a discrete-time setting. The asset price process is specified by the drift and diffusion parameters but we allow them to switch their values according to the regime which is described by a first-order Markov switching model. Then we state the problem to obtain the optimal portfolios under the log mean-variance criteria for each period. The framework can be easily implemented, because it is merely a series of quadratic programming problems which maximize the weighted average of the portfolio return while restricting the weighted average risk at a tolerable level. Here these averages are weighted by the regime inference. One of the advantages in our model is that it is also possible to control the risk-return tradeoff within the standard efficient frontier framework.

Also we apply our model to a sector allocation simulation and the result implies that the model allows us to obtain stabilized regimes when the number of regimes is appropriately chosen. Taking advantage of these regimes, we can improve the performances of the portfolio while keeping the rebalancing frequencies low.

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